POLYGON RECONSTRUCTION FROM MOMENTS USING ARRAY PROCESSING

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Summary

In this paper we prove a set of results showing that the vertices of any simply-connected planar polygonal region can be reconstructed from a finite number of its complex moments using array processing. In particular, we derive and illustrate several new algorithms for the reconstruction of the vertices of simply-connected polygons from moments. These results find applications in a variety of apparently disparate areas such as computerised tomography and inverse potential theory, where in the former it is of interest to estimate the shape of an object from a finite number of its projections; while in the latter, the objective is to extract the shape of a gravitating body from measurements of its exterior logarithmic potentials at a finite number of points. The applications of the algorithms we develop to tomography hence expose a seemingly deep connection between the fields of tomography and array processing. This connection implies that a host of numerical algorithms such as MUSIC, Min-norm, and Prony [11] are now available for application to tomographic reconstruction problems.

Our algorithms are based on the idea that the vertices of a simply-connected polygonal region in the plane are determined by a finite number of its moments. Davis [2] showed, using the Motzkin-Schoenberg (MS) formula [12], that a triangle in the plane is uniquely determined by its moments of up to order 3. In the process of proving this result, Davis generalised the MS formula to arbitrary n-gons, and in this paper we make use of this result to generalize Davis' triangle result to arbitrary simply-connected polygons. In particular, we have generalised his result using Prony's method [5] to show that the vertices of a simply-connected, n-gon are uniquely determined by its complex moments of up to order $2n-3$. We show that in tomographic terms, this implies that $2n-2$ projections from distinct angles suffice to uniquely determine the vertices of any simply-connected n-gon. This result is an improvement on theoretical results dealing with reconstructability from few projections such as in [3, 6].

Davis generalised the Motzkin-Schoenberg formula, proving the following result:

Theorem 1 *[1] Let $z_1, z_2, \cdots, z_n$ designate the vertices of a polygon P. Then we can find constants $a_1, \cdots, a_n$ depending upon $z_1, z_2, \cdots, z_n$, (and the way they are connected) but independent of h, such that for all h analytic in the closure of P,*

$$\int\int_P h''(z)\, dx\, dy = \sum_{j=1}^{n} a_j h(z_j). \qquad (1)$$

If $r \geq n$ and $z_{n+1}, \cdots, z_r$ are additional points distinct from $z_1, \cdots, z_n$, and if there are constants $b_1, \cdots, b_r$ which depend only upon $z_1, \cdots, z_r$ such that

$$\int\int_P h''(z) dx dy = \sum_{j=1}^{r} b_j h(z_j) \qquad (2)$$

for all h analytic in the closure of P, then

$$b_j = a_j, \quad 1 \leq j \leq n, \qquad (3)$$
$$b_j = 0, \quad n+1 \leq j \leq r. \qquad (4)$$

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Hence, the formula (1) is a *minimal* representation of the integral of h'' over P in terms of discrete values of h. Specifically, the left-hand side of (1) depends *only* on the values of h at the vertices of P and how they are connected; what values h takes at other points in the complex plane are completely irrelevant in this regard. Furthermore, since we can show that each of the a_j is nonzero for a non-degenerate P, the representation (1) for arbitrary $h(z)$'s can not be reduced to one involving $h(z)$ at fewer points.

In the above theorem, by letting (I) $h(z) = z^k$ and (II) $f(x, y)$ be the indicator function over a simply-connected polygonal region P of the plane, we get

$$\int\int_P (z^k)'' \, dx\, dy = \sum_{j=1}^{n} a_j z_j^k \equiv \tau_k \tag{5}$$
$$= k(k-1) \int\int_P f(x, y)\, z^{k-2}\, dx\, dy$$

Hence the numbers τ_k, which we term *weighted complex moments* (**w-complex moments**), with $\tau_0 = \tau_1 = 0$, give measurements of functions of the vertices z_j for every integer $k \geq 0$. We show that by applying Prony's method, the vertices of a simply-connected polygon P are uniquely determined by its w-complex moments τ_k up through order $2n - 1$. (Or equivalently, the *simple (harmonic) complex moments* or **s-complex moments** c_k defined by

$$c_k = \int\int_P f(x, y)\, z^k\, dx\, dy \tag{6}$$

up through order $2n - 3$.) In addition, we discuss the remarkable fact that these moments (or in fact *all* w-complex moments of a polygon) are in general not sufficient to uniquely specify the interior of the polygon, even though they *do* uniquely specify the vertices. See Figure 1.

The explicit connection between the above and array processing emerges when we consider the general array processing problem of estimating the unknowns c_j and z_j from the measured signals y_k given as follows

$$y_k = \sum_{j=1}^{n} c_j z_j^k + v_k, \quad k = 0, \cdots, N-1 \tag{7}$$

where, z_j denotes an unknown source, c_j denotes an unknown complex amplitude, and v_k denotes (complex) white noise. Now assume that noisy estimates $\hat{\tau}_k$ of the w-complex moments of a simply-connected n-gon are given:

$$\hat{\tau}_k = \sum_{j=1}^{n} a_j z_j^k + w_k. \tag{8}$$

By comparing this measurement equation to (7), we can see that they have exactly the same form; whereby a vertex of the polygon can be interpreted as a radiating source whose corresponding (complex) amplitude shows how it is connected to the other vertices of the polygon. The general formulation of the array processing problem is therefore nearly the same as the formulation of the reconstruction problem of binary polygonal objects from noisy measurements of their w-complex moments. The main difference is that the coefficients a_j are *not* independent variables but are, in fact, deterministic functions of z_j and the order in which they are connected. Nevertheless, if we treat the a_j as independent unknowns, we *can* directly apply array processing methods and then check to see if the a_j so-determined are in fact consistent with one of the finite number of polygons with vertices given by the extracted values z_j.

A novel application of the concepts and algorithms discussed above can be found in the field of tomographic reconstruction. It is easily shown that the moments τ_k are complex linear combinations of moments of the underlying image. In addition, the Radon transform of the image $f(x, y)$ defined by:

$$g(t, \theta) = \int\int_O f(x, y)\delta(t - \omega \cdot [x, y]^T) dx dy, \tag{9}$$

satisfies an elementary but extremely useful property that if $F(t)$ is *any* square integrable function, then the following relation holds true:

$$\int_{-T}^{T} g(t, \theta) F(t) dt = \int\int_O f(x, y) F(\omega \cdot [x, y]^T) dx dy. \tag{10}$$

where T denotes the maximal support value of the set O in the direction θ defined by $T = \max_O(x\cos(\theta) + y\sin(\theta))$. By considering $F(t) = e^{-it}$, the celebrated *Projection Slice Theorem* [4] is obtained. By letting $F(t) = t^k$ and expanding the right-hand side of (10) using the binomial theorem, we obtain

$$H^{(k)}(\theta) = \int_{-T}^{T} g(t, \theta)\, t^k\, dt \tag{11}$$
$$= \sum_{j=0}^{k} \binom{k}{j} \cos^{k-j}(\theta)\, \sin^j(\theta)\, \mu_{k-j,j} \tag{12}$$

where

$$\mu_{pq} = \int\int_P f(x, y) x^p y^q\, dx\, dy. \tag{13}$$

which shows that the k^{th} order moment of the projection at angle θ is a linear combination of the k^{th} order moments of the image [8, 10, 9]. Furthermore, we have proved the following [8, 10, 7]:

Proposition 1 *Given line integral projections of $f(x, y)$ at m different angles θ_j in $[0, \pi)$, one can uniquely determine the first m moment vectors $\mu^{(j)}$ (set of all moments of order $j = p + q$), $0 \leq j < m$ of $f(x, y)$. This can be done using only the first m geometric moments $H^{(k)}(\theta_j)$, $0 \leq k < m$ of the projections. Furthermore, moments of $f(x, y)$ of higher order* cannot *be uniquely determined from m projections.*

Hence, invoking this "moment-property" of the Radon transform, we compute Maximum Likelihood estimates of the w-complex moments of the underlying polygon from noisy projections, and having these, we directly apply array processing algorithms to recover the vertices of the polygon. In particular, we concentrate on some important differences between this tomographic scenario and the standard array processing scenario. We show that the tomographic polygonal reconstruction problem from moments requires the development of specialized array processing algorithms which exploit the particular structure of the noise statistics arising in the estimation of the w-complex moments.

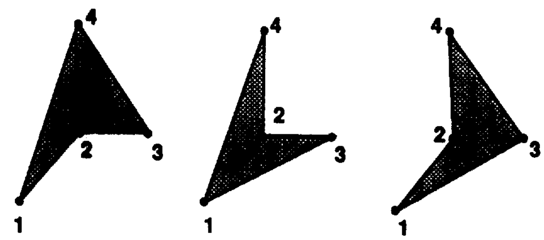

Figure 1: Three distinct regions corresponding to the same vertices

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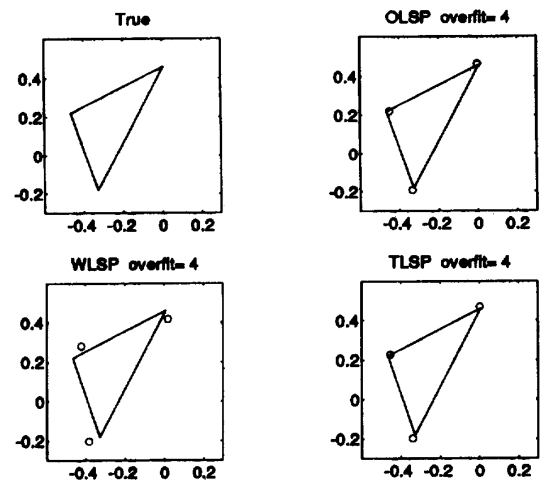

Figure 2: Sample reconstructions at SNR=23.9 dB solid: actual, circles: reconstructed. Estimated τ_k for $0 \leq k \leq 9$ used (i.e. overfit=4.)

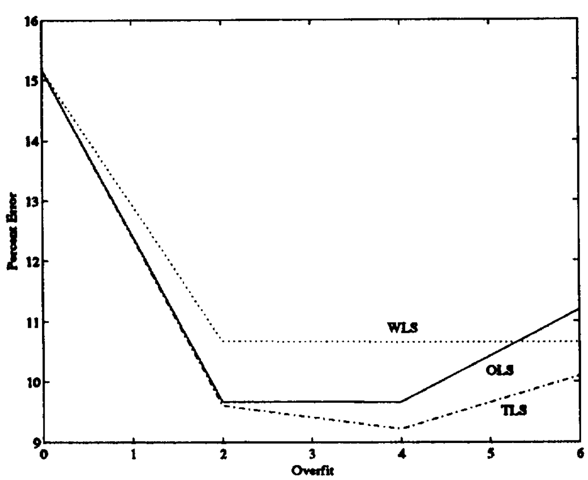

Figure 3: Overlayed performance curves for Ordinary-, Weighted-, and Total-least squares Prony techniques using noisy projections at SNR=23.9 dB