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# Hedging Structured Concepts

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Prediction With  
Expert Advice

Structured Concepts

Component Hedge

Conclusion

- Prediction With Expert Advice

Hedge algorithm

- Structured Concepts

SC + Hedge  $\Rightarrow$  range factor problem

- Component Hedge

SC + CH  $\Rightarrow$  range factor problem solved

- Conclusion

# Prediction with Expert Advice

Prediction With  
Expert Advice

▷ Prediction with  
Expert Advice

The Hedge  
Algorithm

Structured Concepts

Component Hedge

Conclusion

## □ Setting

- Several sources of predictions (experts)
- Choose an expert each trial (randomised)
- Incur loss of the selected expert (0/1)
- Observe loss of all experts (full information)

## □ Goal

- Cumulative loss close to the best expert
- Efficient algorithm

# The Hedge Algorithm (Freund & Schapire 1997)

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Prediction with  
Expert Advice

▷ The Hedge  
Algorithm

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Component Hedge

Conclusion

- Maintains uncertainty as a distribution  $w_t$  on  $n$  experts  
 $w_1$  is uniform
- For each trial  $t = 1, 2, \dots$ 
  - Select expert  $i$  with probability  $w_{t,i}$
  - Receive loss vector  $\ell_t \in [0, 1]^n$ , incur loss  $\ell_{t,i}$
  - Expected loss  $w_t \cdot \ell_t$
  - Update  $w_{t+1,i} \propto w_{t,i} \beta^{\ell_{t,i}}$
- With  $\ell^H = \sum_{t=1}^T w_t \cdot \ell_t$  and  $\ell^* = \min_i \sum_{t=1}^T \ell_{t,i}$ ,

$$\ell^H - \ell^* \leq \sqrt{2\ell^* \ln n} + \ln n$$

# Structured Concepts

Prediction With  
Expert Advice

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Structured Concepts

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▷ Structured  
Concepts

Prediction with  
Structured Concepts  
Expanded Hedge

Component Hedge

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Conclusion

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- Concepts composed of components

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concept	component
set	element
permutation	assignment
bipartite matchings	edges
spanning trees	edges
paths	edges

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# Prediction with Structured Concepts

Prediction With  
Expert Advice

Structured Concepts

Structured Concepts  
Prediction with  
Structured

▷ Concepts

Expanded Hedge

Component Hedge

Conclusion

- Goal: on-line prediction with “combinatorial experts”
  - Route planning: shortest path
  - Media multicasting: directed spanning trees
- Loss of concept is sum of losses of its components
- Helps: losses of concepts highly related
- Hurts: combinatorial explosion (many concepts)

# Expanded Hedge (EH)

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Structured Concepts  
Prediction with  
Structured Concepts

▷ Expanded Hedge

Component Hedge

Conclusion

- Treat each structured concept as an expert
- Run Hedge algorithm
- Consider size  $k$  subsets of  $n$  elements
  - Component loss in  $[0, 1]$ , so concept loss in  $[0, k]$ .
  - Number of concepts  $\binom{n}{k} \approx n^k$ .
  - Regret bound

$$\ell^{\text{EH}} - \ell^* \leq \sqrt{2\ell^* k k \ln n} + k k \ln n$$

- But lower bound has  $k \ln n$ . Range factor problem

# Usages

Prediction With  
Expert Advice

Structured Concepts

Component Hedge

▷ Usages

Usages Example

Expanded Hedge

Component Hedge

Component Hedge II

Implementation

Lower Bounds

Conclusion

- Identify concepts with incidence vectors
- Loss of  $C$  is  $C \cdot \ell$  (with  $\ell$  component losses)
- Randomly select a concept  $C$  with probability  $W_C$
- Expected loss is

$$\sum_C W_C (C \cdot \ell) = \underbrace{\left( \sum_C W_C C \right)}_{\text{usage of } W} \cdot \ell$$

- Only the *usage* (i.e. mean concept) matters
- Set of usages is the convex hull of concepts

# Usages Example

Prediction With  
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Usages

▷ Usages Example

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- Sets of 2 out of 4 elements

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

- The usage of the distribution  $(.3, .3, .2, .1, .1, 0)$  on sets

$$.3 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + .3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + .2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + .1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + .1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} .8 \\ .5 \\ .4 \\ .3 \end{pmatrix}$$

# Expanded Hedge (EH)

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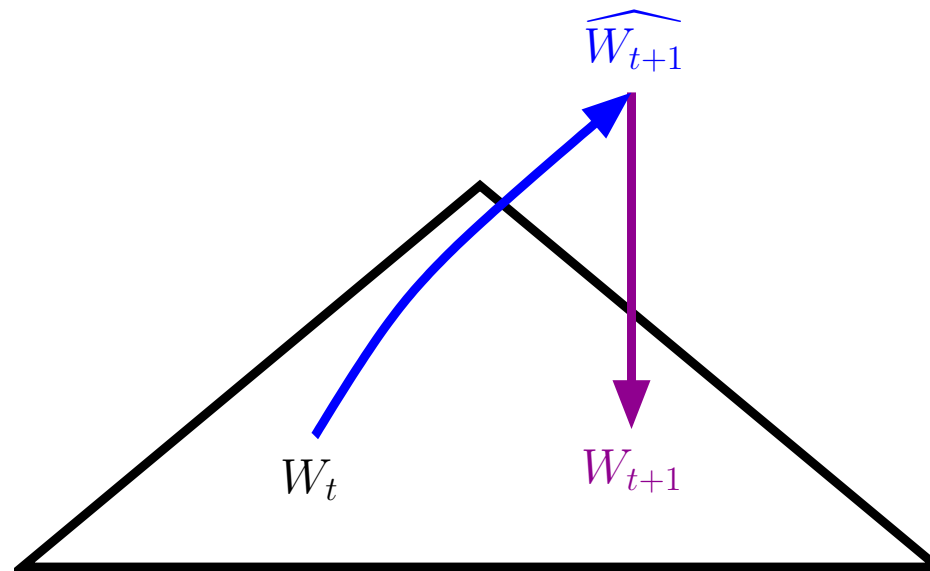
Lower Bounds

Conclusion

## Two-step EH update

$$\widehat{W}_{t+1} = \operatorname{argmin}_W \Delta(W \| W_t) + \sum_C W_C (C \cdot \ell_t)$$

$$W_{t+1} = \operatorname{argmin}_{W \text{ a p.d.}} \Delta(W \| \widehat{W}_{t+1})$$



$$\Delta(x \| y) = \sum_i x_i \ln \frac{x_i}{y_i} - x_i + y_i$$

# Component Hedge(CH)

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Usages Example

Expanded Hedge

Component  
▷ Hedge

Component Hedge II

Implementation

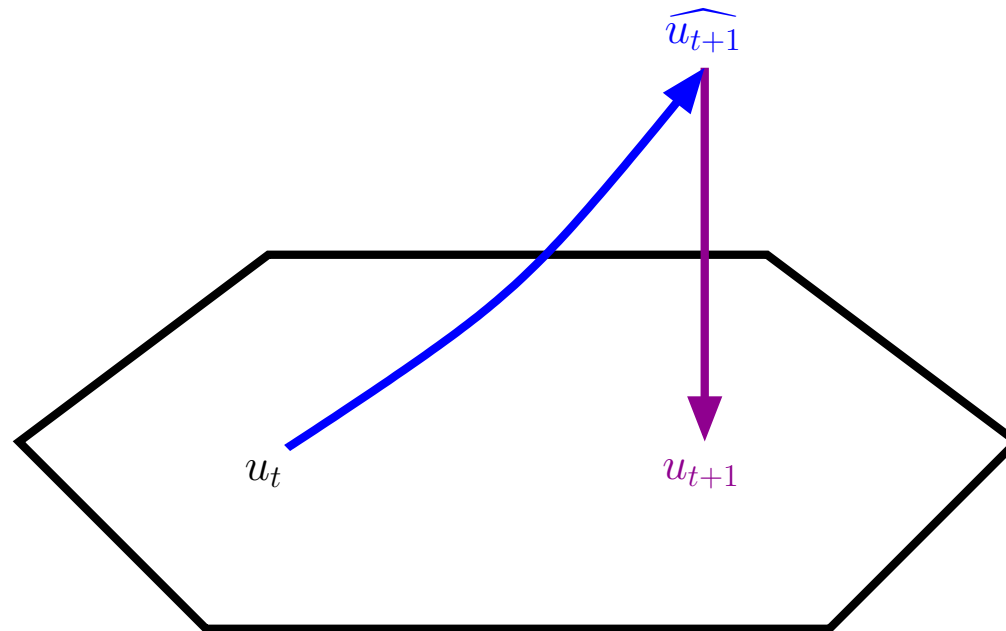
Lower Bounds

Conclusion

Idea: do the same trick on the level of usages

$$\widehat{u}_{t+1} = \operatorname{argmin}_u \Delta(u \| u_t) + u \cdot \ell_t$$

$$u_{t+1} = \operatorname{argmin}_{u \text{ a usage}} \Delta(u \| \widehat{u}_{t+1})$$



# Component Hedge II

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Usages

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Component Hedge

▷ Component  
Hedge II

Implementation

Lower Bounds

Conclusion

□ Let  $u_1$  be the usage of the uniform distribution

□ For trial  $t = 1, 2, \dots$

– Decompose  $u_t = \sum_i \alpha_i C_i$

– Sample  $C_i$  with probability  $\alpha_i$

– Expected loss  $u_t \cdot \ell_t$

– Update and relative entropy projection

□ Regret has no range factor. E.g. for  $k$ -of- $n$  sets

$$\ell^{\text{CH}} - \ell^* \leq \sqrt{2\ell^* k \ln n} + k \ln n$$

# Implementation

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▷ Implementation

Lower Bounds

Conclusion

- Usage vectors  $u_t$  are small
- No closed form for relative entropy projection

$$u_{t+1} = \underset{u \text{ a usage}}{\operatorname{argmin}} \Delta(u \| \widehat{u}_{t+1})$$

- The usage polytope is the convex hull of exponentially many concepts. Fortunately, it can often be represented by polynomially many linear inequalities. E.g. Birkhoff and flow polytope.
- Idea: iteratively reestablish most violated constraint
- Known as Sinkhorn balancing for permutations

# Lower Bounds

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▷ Lower Bounds

Conclusion

- CH is optimal: we have matching lower bounds for sets, permutations, bipartite matchings, spanning trees and paths.
- In each case, reduction from the basic expert case.

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▷ Philosophy

## □ Uncertainty

- EH: Probability distribution on concepts
- CH: Convex combination of concepts

## □ Relative entropy regularisation seems universal

- Possible to incorporate constraints into divergence
- But RE works in all cases