CS101 Larrabee Autumn '02 Name: ______________________

Assessment Quiz, Sept 20, 2002

For each question, give the most exact, closed form answer you know. Also try to give the best asymptotic (Big "O") notation for the answer. Don't spend too much effort on this, it is for background evaluation only.

1. $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$ and is $O(k^2)$.

2. for positive $j < k$, $\sum_{i=j}^{k} i = \sum_{i=0}^{k} i - \sum_{i=0}^{j-1} i = \frac{k(k+1)}{2} - \frac{(j-1)j}{2}$ and is $O(k^2 - j^2)$

3. $\log_2(4i^3) = \log_2(4) + \log_2(i^3) = 2 + 3\log_2(i)$ and is $O(\log i)$.

4. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ using factorials (see also Stirling's approximation). For fixed k this is $O(n^k)$.

5. What is the maximum number of edges in a simple undirected graph of n nodes (i.e. the complete graph on n nodes)?

`Each node is connected to` $n-1$ `other nodes, but this counts each edge twice. Therefore there are` $n(n-1)/2$ `nodes (which is` $O(n^2)$`) in the complete graph on` n `nodes.`

6. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$. This one you should all learn immediately, and is $O(2^k)$.

7. Assume that:

- A complete binary tree of height 0 has 1 node, and
- A complete binary tree of height $h > 0$ consists of a root connected to two complete binary trees of height $h - 1$.

Describe how to prove *by induction* that complete binary trees of height h have $2^{h+1} - 1$ nodes. I am interested in rough sketch of what you would do rather than a formal proof.

Statement of theorem to prove: `Theorem: Complete binary trees of height` $h \geq 0$ `have exactly` $2^{h+1} - 1$ `nodes.`

Statement of proof method: `Proof by induction.`

Identify the statement you are inducting over: `Inductive Hypothesis: all complete binary trees of height` k `have` $2^{k+1} - 1$ `nodes.`

Show the inductive hypothesis holds for a particular value. `Base Case: All binary trees of height 0 have 1 node (given), which equals` $2^{0+1} - 1$`.` Here the base case value is 0.

Show that if the inductive hypothesis happens to hold for any particular j greater than or equal to the base case value, then the inductive hypothesis for $j + 1$ also holds.

`Inductive Step: Assume the inductive hypothesis for some` $j \geq 0$ `holds, i.e. that all complete binary trees of height` j `have` $2^{j+1}-1$ `nodes. Any binary tree of height` $j+1$ `contains a root plus two binary trees of height` j `(given). Thus (using the inductive hypothesis for` j`) the height` $j+1$ `binary tree contains` $1 + 2 \cdot (2^{k+1} - 1) = 2^{k+2} - 1$ `nodes.`

Conclude that the inductive hypothesis holds for all k greater than or equal to the base value: `Therefore, for all` $k \geq 0$`, all complete binary trees of height` k `have` $2^{k+1} - 1$ `nodes.`

As this is the Theorem we wanted to prove, we are done. Sometimes the inductive hypothesis is not the same as the statement of the Theorem, and a little additional work is required. (Note that the inductive hypothesis is parameterized by k, and I used j in the inductive step. I think different letters make things less confusing.)