Recap Find maxima and minima of $f(x,y)$

Step 1: Find critical points

Step 2: Second derivative test.

Dependence of the maximum value of ~~$f(x,y)$~~ $F(x,y;\alpha,\beta)$ on parameters (α,β)

Notation: $F(x,y;\alpha,\beta)$

x, y: variables (prices in markets A and B)

α, β: parameters (market conditions of A and B)

F: Total profit in the two markets.

Ex: $F(x,y;\alpha,\beta) = (\alpha+y-x)(x-2)+(\beta+x-2y)(y-2)$

When $(\alpha,\beta)=(4,6)$

$(x^*,y^*)=(8,6)$ is a maximum

$F(x^*,y^*;4,6)=20$

Back to the general case.

Let (x^*, y^*) be the maximum of $F(x, y; \alpha, \beta)$

$$x^* = x^*(\alpha, \beta)$$
$$y^* = y^*(\alpha, \beta)$$

Let $F^*(\alpha, \beta) = F(x^*(\alpha, \beta), y^*(\alpha, \beta); \alpha, \beta)$

maximum value of $F(x, y; \alpha, \beta)$ at (α, β)

Remark: *) F^* is a function of (α, β).

*) We usually do not have an analytical expression of $F^*(\alpha, \beta)$.

Q: How does $F^*(\alpha, \beta)$ change when (α, β) changes
(for example, to $(4.1, 6)$)

Linear Taylor polynomial:

$$\Delta F^* = F^*(\alpha_0 + \Delta\alpha, \beta_0 + \Delta\beta) - F^*(\alpha_0, \beta_0)$$
$$\approx \left.\frac{\partial F^*}{\partial \alpha}\right|_{(\alpha_0, \beta_0)} \Delta\alpha + \left.\frac{\partial F^*}{\partial \beta}\right|_{(\alpha_0, \beta_0)} \Delta\beta$$

The envelope theorem

$$\frac{\partial F^*}{\partial \alpha} = \frac{\partial F(x^*(\alpha,\beta),\ y^*(\alpha,\beta);\ \alpha,\beta)}{\partial \alpha}$$

$$= \underbrace{\left.\frac{\partial F}{\partial x}\right|_{(x^*,y^*)}}_{=0} \cdot \frac{\partial x^*}{\partial \alpha} + \underbrace{\left.\frac{\partial F}{\partial y}\right|_{(x^*,y^*)}}_{=0} \cdot \frac{\partial y^*}{\partial \alpha} + \left.\frac{\partial F}{\partial \alpha}\right|_{(x^*,y^*)}$$

$$\boxed{\frac{\partial F^*}{\partial \alpha} = \left.\frac{\partial F}{\partial \alpha}\right|_{(x^*,y^*)}}$$

Ex $F(x,y;\alpha,\beta) = (\alpha+y-x)(x-2) + (\beta+x-2y)(y-2)$

When $(\alpha,\beta) = (4,6)$

$(x^*,y^*) = (8,6)$

$F^*(4,6) = 20$

Find an approximation of $F^*(4.1,6)$.

$$\frac{\partial F^*}{\partial \alpha} = \left.\frac{\partial F}{\partial \alpha}\right|_{(x^*, y^*)} = (x^* - 2)$$

$$\left.\frac{\partial F^*}{\partial \alpha}\right|_{(\alpha,\beta)=(4,6)} = 8 - 2 = 6$$

$$\Delta F^* = F^*(4.1, 6) - F^*(4, 6)$$

$$\approx \frac{\partial F^*}{\partial \alpha} \cdot \Delta \alpha = 6 \times 0.1 = 0.6 . \longrightarrow F^*(4.1, 6) \approx 20 + 0.6$$
$$= 20.6 .$$

True $\underline{F^*(4.1, 6) = 20.605}$

<u>Ex</u>. Continue with the example above.

Find an approximation of $F^*(4.1, 5.9)$

$$\frac{\partial F^*}{\partial \beta} = \left.\frac{\partial F}{\partial \beta}\right|_{(x^*, y^*)} = y^* - 2$$

$$\left.\frac{\partial F^*}{\partial \beta}\right|_{(\alpha,\beta)=(4,6)} = 6 - 2 = 4$$

$$\Delta F^* = F^*(4.1,\ 5.9) - F^*(4,\ 6)$$

$$\approx \frac{\partial F^*}{\partial \alpha}\Delta\alpha + \frac{\partial F^*}{\partial \beta}\Delta\beta = 6\times(0.1) + 4\times(-0.1) = 0.2$$

$$\rightarrow \quad F^*(4.1,\ 5.9) \approx 20 + 0.2 = 20.2$$

True $F^*(4.1, 5.9) = 20.2025$

Ex. Direct solution of $x^*(\alpha,\beta)$ and $y^*(\alpha,\beta)$

$$F(x, y; \alpha, \beta) = (\alpha + y - x)(x-2) + (\beta + x - 2y)(y-2)$$

$$F_x = -(x-2) + (\alpha + y - x) + (y-2) = -2x + 2y + \alpha$$

$$F_y = (x-2) - 2(y-2) + (\beta + x - 2y) = 2x - 4y + \beta + 2$$

Find (x^*, y^*)

$$-2x + 2y + \alpha = 0 \qquad (1)$$

$$2x - 4y + \beta + 2 = 0 \qquad (2)$$

$$(1)+(2) \longrightarrow \quad -2y + \alpha + \beta + 2 = 0$$

$$\longrightarrow \quad y = \frac{\alpha + \beta + 2}{2}$$

$(1) \Longrightarrow \quad 2x = 2y + \alpha = 2\alpha + \beta + 2 \longrightarrow x = \dfrac{2\alpha+\beta+2}{2}$

$$\left\{\begin{aligned} x^*(\alpha,\beta) &= \frac{2\alpha+\beta+2}{2} \\ y^*(\alpha,\beta) &= \frac{\alpha+\beta+2}{2} \end{aligned}\right.$$

Ex. $F(x,y;\alpha,\beta) = \left(4 + (\alpha-4)\cdot\dfrac{30}{10+\sqrt{x^2+y^2}} + y - x\right)(x-2)$

$+ (\beta + x - 2y)(y-2)$

when $(\alpha,\beta) = (4,6)$

$(x^*, y^*) = (8, 6)$

$F^*(4,6) = 20$

Find an approximation of $F^*(4.1, 6)$

$$\frac{\partial F^*}{\partial \alpha} = \left.\frac{\partial F}{\partial \alpha}\right|_{(x^*,y^*)} = \frac{30}{10+\sqrt{(x^*)^2+(y^*)^2}}(x^*-2)$$

$$\left.\frac{\partial F^*}{\partial \alpha}\right|_{(\alpha,\beta)=(4,6)} = \frac{30}{10+\sqrt{8^2+6^2}}(8-2) = \frac{30}{10+10}\times 6 = 9$$

$$\Delta F^* = F^*(4.1, 6) - F^*(4, 6)$$

$$\approx \frac{\partial F^*}{\partial \alpha} \Delta \alpha = 9 \times (0.1) = 0.9 \quad \to F^*(4.1, 6) \approx 20 + 0.9 = \underline{20.9}$$

Simulation result

$$F^*(4.1, 6) \approx 20.9051$$

---

§17.7 Lagrange multipliers.

Consider finding maxima / minima of

$$f(x, y) \text{ subject to } \underbrace{g(x, y) = 0}_{\text{constraint.}}$$

<u>Ex</u>. $f(x,y) = 2x^2 + 3y^2$ subject to $\underbrace{x^2+y^2=1}_{x^2+y^2-1=0}$

Recall the case of $f(x)$ <u>with no constraint</u>.

Linear Taylor polynomial

$$f(x) \approx T_1(x) = f(x_0) + f'(x_0)\Delta x, \quad \Delta x = x - x_0$$

x_0 is a critical point of $f(x)$ $\Longleftrightarrow$ $f'(x_0)=0$

$\Longleftrightarrow$ $T_1(x)$ does not vary with Δx.

<u>Back to $f(x,y)$ subject to $g(x,y)=0$.</u>

Linear Taylor polynomials around (x_0, y_0) satisfying $\underline{g(x_0, y_0)=0}$

$$f(x,y) \approx T_1^{(f)}(x,y) = f(x_0,y_0) + f_x \Delta x + f_y \Delta y, \quad \begin{array}{l} \Delta x = x - x_0 \\ \Delta y = y - y_0 \end{array}$$

$$g(x,y) \approx T_1^{(g)}(x,y) = \underbrace{g(x_0,y_0)}_{=0} + g_x \Delta x + g_y \Delta y$$

(x_0, y_0) is a critical point

$\Longleftrightarrow$ $T_1^{(f)}(x,y)$ does not vary with $(\Delta x, \Delta y)$ constrained by $T_1^{(g)}(x,y)=0$

$\Longleftrightarrow$ $f_x \Delta x + f_y \Delta y = 0$ for $(\Delta x, \Delta y)$ satisfying $g_x \Delta x + g_y \Delta y = 0$

$$\Longleftrightarrow \quad \begin{cases} f_x = \lambda_0 g_x \\ f_y = \lambda_0 g_y \end{cases}$$

Consider $F(x,y,\lambda) = f(x,y) - \lambda\, g(x,y)$

$$f_x - \lambda_0 g_x = 0 \longrightarrow F_x\Big|_{(x_0,y_0,\lambda_0)} = 0$$

$$f_y - \lambda_0 g_y = 0 \longrightarrow F_y\Big|_{(x_0,y_0,\lambda_0)} = 0$$

$$g(x_0,y_0) = 0 \longrightarrow F_\lambda\Big|_{(x_0,y_0,\lambda_0)} = 0$$

(x_0, y_0) is a critical point of $f(x,y)$ subject to $g(x,y)=0$

if there exists λ_0 such that

(x_0, y_0, λ_0) is a critical point of $F(x,y,\lambda)$

λ_0 is called a Lagrange multiplier

Ex. Find critical points of

$f(x,y,z) = xyz$ (where $xyz \neq 0$) subject to $3x+y+2z=18$

Consider $F(x,y,z,\lambda) = xyz - \lambda(3x+y+2z-18)$

$F_x = yz - 3\lambda = 0 \qquad (1)$

$F_y = xz - \lambda = 0 \qquad (2)$

$F_z = xy - 2\lambda = 0 \qquad (3)$

$F_\lambda = 3x + y + 2z - 18 = 0$

(1), (2) $\dfrac{yz}{xz} = \dfrac{3\lambda}{\lambda} \rightarrow \dfrac{y}{x} = 3 \rightarrow \boxed{y = 3x}$

(1), (3) $\dfrac{yz}{xy} = \dfrac{3\lambda}{2\lambda} \rightarrow \dfrac{z}{x} = \dfrac{3}{2} \rightarrow \boxed{z = \dfrac{3}{2}x}$

(4) $\rightarrow 3x + 3x + 2\times\left(\frac{3}{2}x\right) - 18 = 0$

$\rightarrow 9x - 18 = 0 \rightarrow \boxed{x = 2}, \boxed{y = 3x = 6}, \boxed{z = \frac{3}{2}x = 3}$

Critical point $(2, 6, 3)$.

EX. Find critical points of $f(x,y)=2x^2+3y^2$ subject to $x^2+y^2=1$

Consider $F(x,y,\lambda)=2x^2+3y^2-\lambda(x^2+y^2-1)$

$$F_x = 4x - 2\lambda x = 0 \quad (1)$$

$$F_y = 6y - 2\lambda y = 0 \quad (2)$$

$$F_\lambda = x^2+y^2-1 = 0 \quad (3)$$

$(1) \rightarrow 2x(2-\lambda)=0 \rightarrow x=0$ or $\lambda=2$

$(2) \rightarrow 2y(3-\lambda)=0 \rightarrow y=0$ or $\lambda=3$

$(3) \rightarrow x^2+y^2=1$

$\cancel{x=0,\ y=\pm1,\ \boxed{\lambda=3}}$

*) Case 1 $\lambda\neq 2$

$\rightarrow x=0 \rightarrow y=\pm 1 \rightarrow \lambda=3$

*) Case 2. $\lambda=2$

$\rightarrow y=0 \rightarrow x=\pm 1$

$x=0,\ y=\pm1,\ \lambda=3$

$x=\pm1,\ y=0,\ \lambda=2$

Critical points of $f(x,y) = 2x^2 + 3y^2$ subject to $x^2+y^2=1$

$(0, 1)$, $(0, -1)$

$(1, 0)$, $(-1, 0)$