<u>Recap</u> Taylor polynomials of $f(x, y)$ around (x_0, y_0)

$$T_1(x, y) = f(x_0, y_0) + \underbrace{f_x}_{f_x(x_0, y_0)} \cdot (x - x_0) + f_y \cdot (y - y_0)$$

$$T_2(x, y) = T_1(x, y) + \frac{1}{2} f_{xx} \cdot (x - x_0)^2 + f_{xy} \cdot (x - x_0)(y - y_0) + \frac{1}{2} f_{yy} \cdot (y - y_0)^2$$

Maxima and minima

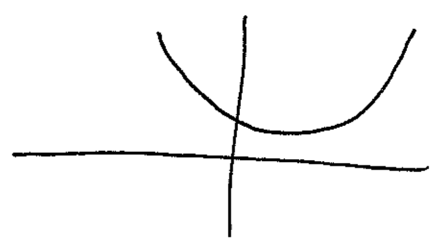
One-variable case $f(x)$

<u>Rule A1</u>: If $x = a$ is a relative maximum (or minimum) then $x = a$ is a critical point.

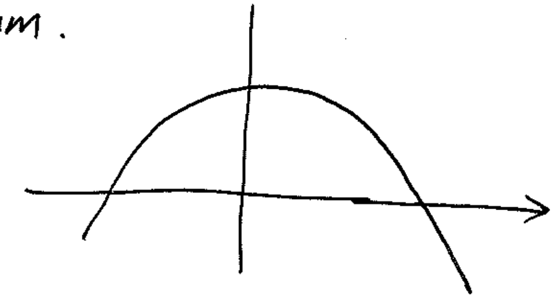
<u>Rule A2</u>: (Second derivative test)

Let $x = a$ be a critical point.

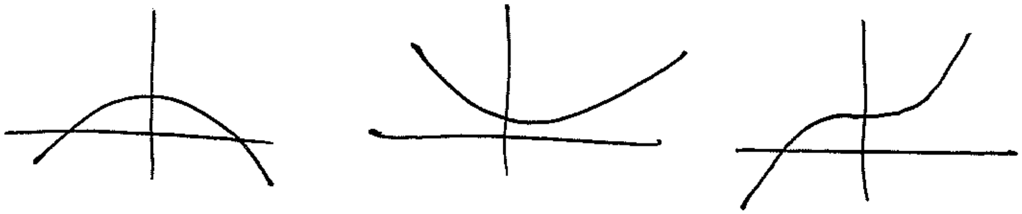
*) $f''(a) > 0 \longrightarrow x = a$ is a relative minimum



*) $f''(a) < 0 \longrightarrow x = a$ is a maximum.



*) $f''(a) = 0 \longrightarrow$ inconclusive.



Behavior of quadratic form

$$F(x,y) = \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2$$

let $D = AC - B^2$, discriminant

*) $D > 0$ and $A > 0$

$$F(x,y)\Big|_{y=\lambda x} \geq 0 \quad \text{for all } \lambda \text{ and all } x$$

$\longrightarrow (0,0)$ is a minimum

*) $D > 0$ and $A < 0$

$$F(x,y)\Big|_{y=\lambda x} \leq 0 \quad \text{for all } \lambda \text{ and all } x$$

$\longrightarrow (0,0)$ is a maximum

*) $D < 0$ $\quad F(x,y)\big|_{y=\lambda x}$ > 0 for some λ

< 0 for some other λ

$\longrightarrow$ $(0,0)$ is a saddle point.

*) $D = 0$ $\longrightarrow$ inconclusive.

Two-variable case $f(x,y)$

Rule 1: If (a,b) is a relative maximum (or minimum) then (a,b) is a critical point. of $f(x,y)$.

Rule 2: (second derivative test). let (a,b) be a critical point.

let $D(a,b) \equiv \left(f_{xx} f_{yy} - (f_{xy})^2 \right)\Big|_{(a,b)}$, discriminant

*) $D > 0$ and $f_{xx} > 0$

$\longrightarrow$ (a,b) is a relative minimum

*) $D > 0$ and $f_{xx} < 0$

$\longrightarrow$ (a,b) is a maximum

*) $D < 0$ $\longrightarrow$ (a,b) is a saddle point.

*) $D = 0$ $\longrightarrow$ inconclusive.

Ex. $f(x,y) = x^3 + y^3 - 3xy$

Find critical points and apply the second derivative test.

Step 1: Find critical points.

$$f_x = 3x^2 - 3y, \quad f_y = 3y^2 - 3x$$

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \longrightarrow \begin{cases} x^2 = y \\ y^2 = x \end{cases} \longrightarrow x^4 = x$$

$$\rightarrow x^3 = 1 \quad \text{or} \quad x = 0$$

$$\rightarrow x = 1 \quad \text{or} \quad x = 0$$

Two critical points: $(0,0)$, $(1,1)$

Step 2: Second derivative test.

$$f_{xx} = 6x, \quad f_{xy} = -3, \quad f_{yy} = 6y$$

$$D(x,y) = (6x)(6y) - (-3)^2 = 36xy - 9$$

At $(0,0)$, $D(0,0) = -9 < 0 \longrightarrow (0,0)$ is a saddle point.

At $(1,1)$, $D(1,1) = 36 - 9 = 27 > 0$

$$f_{xx}\Big|_{(1,1)} = 6x\Big|_{(1,1)} = 6 > 0 \longrightarrow (1,1) \text{ is a relative minimum.}$$

Applications

Maximizing production:

Consider the production function of a product

$$P = f(l,k) = 0.5l^2 - 0.2l^3 + 2k^2 - 0.1k^3$$

Step 1: $f_l = l - 0.6l^2$, $f_k = 4k - 0.3k^2$

$$\begin{cases} l - 0.6l^2 = 0 \\ 4k - 0.3k^2 = 0 \end{cases} \longrightarrow \begin{cases} l(1-0.6l) = 0 \\ 4k\left(1 - \frac{0.3}{4}k\right) = 0 \end{cases} \longrightarrow \begin{array}{l} l = 0 \text{ or } l = \frac{1}{0.6} = \frac{5}{3} \\ k = 0 \text{ or } k = \frac{4}{0.3} = \frac{40}{3} \end{array}$$

Four critical points: $(0,0)$, $\left(0, \frac{40}{3}\right)$, $\left(\frac{5}{3}, 0\right)$, $\left(\frac{5}{3}, \frac{40}{3}\right)$

Step 2: $f_{ll} = 1 - 1.2l$, $f_{lk} = 0$, $f_{kk} = 4 - 0.6k$

$$D(l,k) = f_{ll} \cdot f_{kk} - (f_{lk})^2 = (1-1.2l)(4-0.6k)$$

At $(0,0) \Longrightarrow D(0,0) = 1 \times 4 = 4 > 0$, $f_{ll} = 1 > 0$

$\longrightarrow (0,0)$ is a minimum

$A\left(0, \frac{40}{3}\right)$, $D\left(0, \frac{40}{3}\right) = 1 \times \left(4 - 0.6 \times \frac{40}{3}\right) = 1 \times (4-8) = -4 < 0$

$\longrightarrow \left(0, \frac{40}{3}\right)$ is a saddle point.

At $\left(\frac{5}{3}, 0\right)$, $D\left(\frac{5}{3}, 0\right) = \left(1 - 1.2 \times \frac{5}{3}\right) \times 4 = -1 \times 4 = -4 < 0$

$\rightarrow$ $\left(\frac{5}{3}, 0\right)$ is a saddle point.

At $\left(\frac{5}{3}, \frac{40}{3}\right)$, $D\left(\frac{5}{3}, \frac{40}{3}\right) = \left(1 - 1.2 \times \frac{5}{3}\right)\left(4 - 0.6 \times \frac{40}{3}\right) = (1-2) \times (4-8)$

$= (-1) \times (-4) = 4 > 0$

$f_{\ell\ell} = 1 - 1.2 \times \frac{5}{3} = -1 < 0$

$\rightarrow$ $\left(\frac{5}{4}, \frac{40}{3}\right)$ is a maximum.

Profit maximization (Two products A and B).

A candy company produces two types of candy A and B, respectively with costs $C_A = 2/\ell b$ and $C_B = 4/\ell b$

The joint demand functions are

$$q_A = 400\,(P_B - P_A)$$

$$q_B = 400\,(8 + P_A - 2P_B)$$

P_A, P_B: prices of A and B

q_A, q_B: quantities sold of A and B.

Profit as a function prices

$$P(P_A, P_B) = q_A(P_A-2) + q_B(P_B-4)$$

$$= 400\left[(P_B-P_A)(P_A-2) + (8+P_A-2P_B)(P_B-4)\right]$$

Q: How should the company price A and B ?

Step 1:

$$\frac{\partial P}{\partial P_A} = 400\left[-(P_A-2)+(P_B-P_A)+(P_B-4)\right]$$

$$= 400\left[-2P_A+2P_B-2\right]$$

$$\frac{\partial P}{\partial P_B} = 400\left[(P_A-2)-2(P_B-4)+(8+P_A-2P_B)\right]$$

$$= 400\left[2P_A-4P_B+14\right]$$

$$\begin{cases} -2P_A+2P_B-2=0 \\ 2P_A-4P_B+14=0 \end{cases} \longrightarrow \begin{cases} P_A = P_B-1 \\ P_A-2P_B+7=0 \end{cases}$$

$$\longrightarrow (P_B-1)-2P_B+7=0$$

$$\longrightarrow -P_B+6=0 \longrightarrow P_B=6$$

$$\longrightarrow P_A = P_B-1 = 6-1 = 5$$

$$\boxed{\begin{aligned} P_A &= 5 \\ P_B &= 6 \end{aligned}}$$

Step 2.
$$\frac{\partial^2 P}{\partial P_A^2} = 400 \times [-2] < 0$$

$$\frac{\partial^2 P}{\partial P_A \partial P_B} = 400 \times [2]$$

$$\frac{\partial^2 P}{\partial P_B^2} = 400 \times [-4]$$

$$D = \frac{\partial^2 P}{\partial P_A^2} \cdot \frac{\partial^2 P}{\partial P_B^2} - \left(\frac{\partial^2 P}{\partial P_A \partial P_B}\right)^2 = (400)^2\left((-2)(-4) - (2)^2\right)$$

$$= (400)^2 \times (8-4) > 0$$

$\longrightarrow (5,6)$ is a maximum.

Profit maximization (two markets A and B)

A company produces a drug with cost $C=2$ and sells it in two markets A and B

The joint demand functions are

$$q_A = 400(4 + p_B - p_A)$$

$$q_B = 400(6 + p_A - 2p_B)$$

p_A, p_B: prices of the drug, respectively in markets A and B.

q_A, q_B: quantities sold in markets A and B.

Profit as a function of prices.

$$P(p_A, p_B) = q_A(p_A - 2) + q_B(p_B - 2)$$

$$= 400\left[(4 + p_B - p_A)(p_A - 2) + (6 + p_A - 2p_B)(p_B - 2)\right]$$

Q. How should the company price in markets A and B?

Step 1.

$$\frac{\partial P}{\partial p_A} = 400\left[-(p_A - 2) + (4 + p_B - p_A) + (p_B - 2)\right]$$

$$= 400\left[-2p_A + 2p_B + 4\right]$$

$$\frac{\partial P}{\partial p_B} = 400\left[(p_A - 2) - 2(p_B - 2) + (6 + p_A - 2p_B)\right]$$

$$= 400\left[2p_A - 4p_B + 8\right]$$

$$\begin{cases} -2P_A + 2P_B + 4 = 0 \\ 2P_A - 4P_B + 8 = 0 \end{cases} \longrightarrow \begin{cases} P_A = P_B + 2 \\ P_A - 2P_B + 4 = 0 \end{cases}$$

$$\rightarrow (P_B + 2) - 2P_B + 4 = 0$$

$$\rightarrow -P_B + 6 = 0 \rightarrow P_B = 6$$

$$\rightarrow P_A = P_B + 2 = 8$$

$$\boxed{\begin{matrix} P_A = 8 \\ P_B = 6 \end{matrix}}$$

Step2:

$$\frac{\partial^2 P}{\partial P_A^2} = 400 \times [-2] < 0$$

$$\frac{\partial^2 P}{\partial P_A \partial P_B} = 400 \times [2]$$

$$\frac{\partial^2 P}{\partial P_B^2} = 400 \times [-4]$$

$$D = \frac{\partial^2 P}{\partial P_A^2} \cdot \frac{\partial^2 P}{\partial P_B^2} - \left(\frac{\partial^2 P}{\partial P_A \partial P_B} \right)^2 = (400)^2 \left[(-2)(-4) - 2^2 \right]$$

$$= (400)^2 (8 - 4) > 0$$

$\rightarrow (8, 6)$ is a maximum.

We view this problem from a different angle.

Let $x = P_A$, $y = P_B$, $\alpha = 4$, $\beta = 6$

We write the profit as

$$F(x, y; \alpha, \beta) = (\alpha + y - x)(x-2) + (\beta + x - 2y)(y-2)$$

When $(\alpha, \beta) = (4, 6)$

$(x^*, y^*) = (8, 6)$ is a maximum.

x, y are variables.

α, β are parameters.

$$F(x^*, y^*, 4, 6) = 20$$

Observation: (x^*, y^*) varies with (α, β)

$$x^* = x^*(\alpha, \beta)$$

$$y^* = y^*(\alpha, \beta)$$

Maximum profit at market condition (α, β)

$$F^*(\alpha, \beta) = F(x^*(\alpha, \beta), y^*(\alpha, \beta); \alpha, \beta)$$

$$F^*(\alpha,\beta)\Big|_{(\alpha,\beta)=(4,6)} = 20$$

Suppose the market condition changes to

$$\alpha = 4.1, \quad \beta = 6$$

Q: How does the maximum profit $F^*(\alpha,\beta)$ change accordingly?