Recap Applications of partial derivatives

*) Joint cost function $C(x, y)$

*) Production function $P(l, k)$

*) Competitive and complementary products

$$q_A = f(p_A, p_B)$$

$$\frac{\partial q_A}{\partial p_A} < 0 \quad \text{always true}$$

$$\frac{\partial q_A}{\partial p_B} > 0 \quad \text{Competitive}$$

$$\frac{\partial q_A}{\partial p_B} < 0 \quad \text{Complementary}$$

Taylor polynomials

One-variable case $f(t)$ around t_0

$$T_1(t) = f(t_0) + f'(t_0)(t - t_0)$$
$$T_2(t) = T_1(t) + \frac{1}{2} f''(t_0)(t - t_0)^2$$

Two-variable case $F(x,y)$ around (x_0, y_0)

$$T_1(x,y) = C_{0,0} + C_{0,1}(x-x_0) + C_{1,0}(y-y_0)$$

Set $T_1(x_0,y_0) = F(x_0,y_0)$

$$\underset{C_{0,0}}{\underset{\|}{T_1(x_0,y_0)}} = F(x_0,y_0)$$

$$\underset{C_{0,1}}{\underset{\|}{(T_1)_x(x_0,y_0)}} = F_x(x_0,y_0)$$

$$\underset{C_{1,0}}{\underset{\|}{(T_1)_y(x_0,y_0)}} = F_y(x_0,y_0)$$

$$\boxed{T_1(x,y) = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)}$$

Higher partial derivatives. $f(x,y)$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}$$

The quadratic Taylor polynomial in two variables. $F(x,y)$

$$T_2(x,y) = T_1(x,y) + C_{2,0}(x-x_0)^2 + C_{1,1}(x-x_0)(y-y_0) + C_{0,2}(y-y_0)^2$$

Set $(T_2)_{xx}(x_0,y_0) = F_{xx}(x_0,y_0)$

$\quad \| \quad 2C_{2,0}$

$(T_2)_{yy}(x_0,y_0) = F_{yy}(x_0,y_0)$

$\quad \| \quad 2C_{0,2}$

$(T_2)_{xy}(x_0,y_0) = F_{xy}(x_0,y_0)$

$\quad \| \quad C_{1,1}$

$$\boxed{\begin{aligned} T_2(x,y) = T_1(x,y) &+ \tfrac{1}{2}F_{xx}(x_0,y_0)(x-x_0)^2 + F_{xy}(x_0,y_0)(x-x_0)(y-y_0) \\ &+ \tfrac{1}{2}F_{yy}(x_0,y_0)(y-y_0)^2 \end{aligned}}$$

Ex $F(x,y) = e^{x+2y}$

Find $T_1(x,y)$ and $T_2(x,y)$ around $(0,0)$

$F(0,0) = 1$

$F_x = e^{x+2y}$, $F_x(0,0) = 1$

$F_y = e^{x+2y} \cdot 2$, $F_y(0,0) = 2$

$F_{xx} = e^{x+2y}$, $F_{xx}(0,0) = 1$

$F_{xy} = e^{x+2y} \cdot 2$, $F_{xy}(0,0) = 2$

$F_{yy} = e^{x+2y} \cdot 2^2$, $F_{yy}(0,0) = 4$

$T_1(x,y) = 1 + x + 2y$

$T_2(x,y) = 1 + x + 2y + \frac{1}{2}x^2 + 2xy + 2y^2$

Ex. Use $T_1(x,y)$ and $T_2(x,y)$ to estimate $F(0.1, 0.05)$

$T_1(0.1, 0.05) = 1 + 0.1 + 2 \times 0.05 = 1.2$

$T_2(0.1, 0.05) = T_1(0.1, 0.05) + \frac{1}{2}(0.1)^2 + 2 \times (0.1)(0.05) + 2(0.05)^2$

$= 1.2 + 0.005 + 0.01 + 0.005 = 1.22$

True $F(0.1, 0.05) = e^{0.1 + 2 \times 0.05} = 1.22140 \cdots$

§17.6 Maxima and minima of functions of two variables.

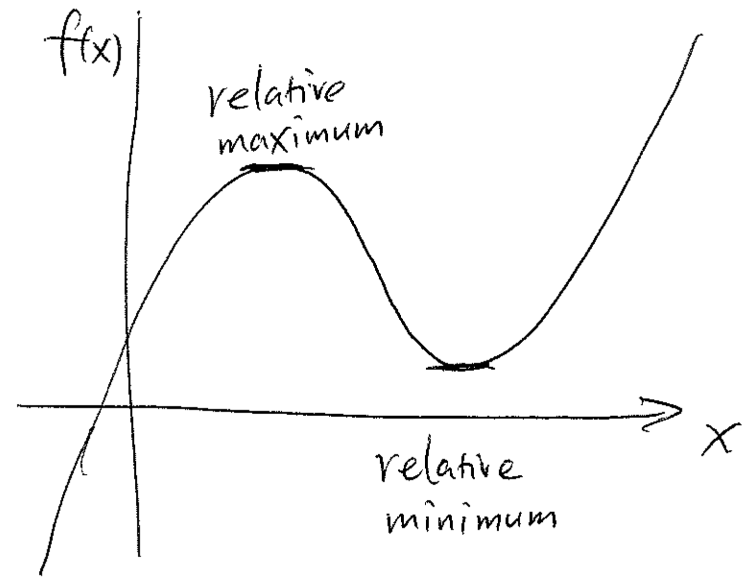
The one-variable case $f(x)$

$f(x)$ has a relative maximum at $x=a$ if

$$f(a) \geq f(x) \quad \text{for all } x \text{ near } a$$

relative minimum

if $f(a) \leq f(x)$ for all x near a



Critical point:

$x=a$ is called a critical point of $f(x)$ if

$$f'(a)=0.$$

Rule A1: If $x=a$ is a relative maximum (or minimum) then $x=a$ is a critical point

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \begin{matrix} \to <0 \text{ for } h>0 \\ \to >0 \text{ for } h<0 \end{matrix}$$

$$=0.$$

Rule A2 Let $x=a$ be a critical point of $f(x)$.

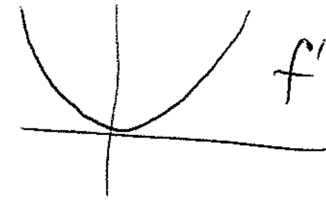
(a candidate for relative maximum or relative minimum)

If $f''(a) > 0$, then $x=a$ is a relative minimum.

If $f''(a) < 0$, then $x=a$ is a relative maximum.

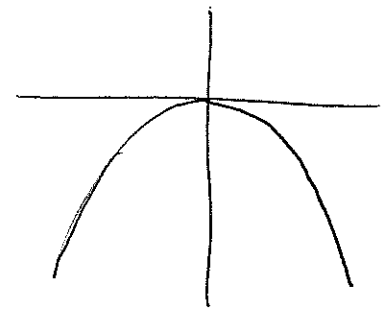
$$T_2(x) = f(a) + \underbrace{f'(a)}_{=0}(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

parabola



$f''(a) > 0$

If $f''(a) = 0$, then it is inconclusive.



$f''(a) < 0$

The two-variable case $f(x,y)$

$f(x,y)$ has a relative maximum at (a,b) if

$f(a,b) \geq f(x,y)$ for all (x,y) near (a,b).

Critical point: (a,b) is called a critical point of $f(x,y)$ if

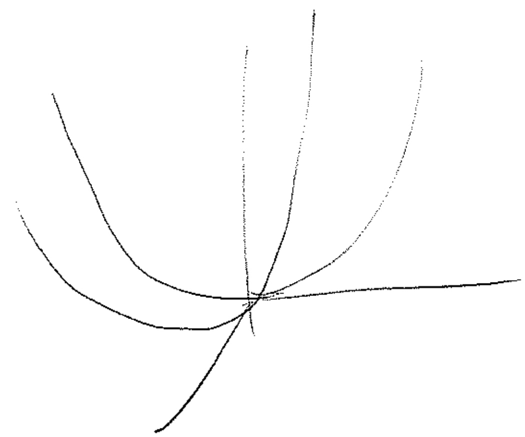
$f_x(a,b) = 0$ and $f_y(a,b) = 0$

Rule 1: If (a,b) is a relative maximum (or minimum) then (a,b) is a critical point.

$\rightarrow$ $x=a$ is a relative maximum of $f(x,b)$

$\rightarrow$ $f_x(a,b)=0$.

$f_y(a,b)=0$.



$\boxed{\text{How about Rule 2 ?}}$

Ex. $f(x,y)=\frac{x^2}{2}+y^2$

$(0,0)$ is a relative minimum.

$f_{xx}=1>0$, $f_{yy}=2>0$, $f_{xy}=0$.

Ex. $f(x,y)=-\frac{x^2}{2}-y^2$

$(0,0)$ is a relative maximum

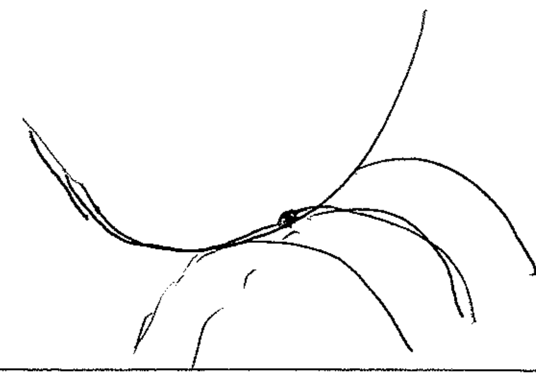
$f_{xx}=-1<0$, $f_{yy}=-2<0$, $f_{xy}=0$.

Ex. $f(x,y) = \frac{x^2}{2} - y^2$

Along $y=0$. $f(x,0) = \frac{x^2}{2}$ has a minimum at $x=0$.

Along $x=0$, $f(0,y) = -y^2$ has a maximum at $y=0$.

$(0,0)$ is a saddle point



$f_{xx} = 1 > 0$,

$f_{yy} = -2 < 0$, $f_{xy} = 0$.

Behavior of quadratic function.

$$Q(\lambda) = \frac{1}{2} A \lambda^2 + B\lambda + \frac{1}{2} C$$

$B^2 - AC < 0$ and $A > 0$

$\longrightarrow$ $Q(\lambda) > 0$ for all λ.

$B^2 - AC < 0$ and $A < 0$.

$\longrightarrow$ $Q(\lambda) < 0$ for all λ.

$B^2 - AC > 0 \longrightarrow Q(\lambda)$ > 0 for some λ
< 0 for some other λ.

Behavior of

$$\boxed{F(x,y) = \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2}$$

Along the direction $y = \lambda x$

$$F(x, \lambda x) = x^2 \left[\underline{\frac{1}{2}C\lambda^2 + B\lambda + \frac{1}{2}A} \right]$$

$\boxed{AC - B^2 > 0 \quad \text{and} \quad A > 0.}$

$\rightarrow$ $F(x, \lambda x) > 0$ for all λ and $x \neq 0$

$\rightarrow$ $(0,0)$ is a <u>relative minimum</u>

$\boxed{AC - B^2 > 0 \quad \text{and} \quad A < 0}$

$\rightarrow$ $F(x, \lambda x) < 0$ for all λ and $x \neq 0$

$\rightarrow$ $(0,0)$ is a <u>relative maximum</u>

$\boxed{AC - B^2 < 0}$ $\rightarrow$ $F(x, \lambda x)$ has a minimum for some λ.

has a maximum for some other λ.

$\rightarrow$ $(0,0)$ is a <u>saddle point</u>.

$\boxed{\text{Rule 2}}$ let (a,b) be a critical point of $f(x,y)$

$$T_2(x,y) = f(a,b) + \underbrace{f_x(a,b)}_{=0}(x-a) + \underbrace{f_y(a,b)}_{=0}(y-b)$$

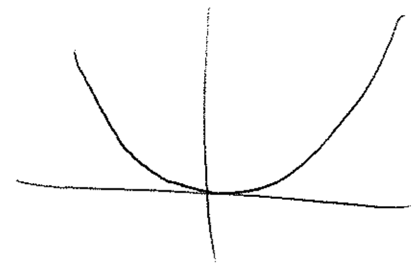
$$+\frac{1}{2}f_{xx}(x-a)^2 + f_{xy}(x-a)(y-b) + \frac{1}{2}f_{yy}(y-b)^2$$

Compare it with $\frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2$

Let $\boxed{D = f_{xx}f_{yy} - (f_{xy})^2}$

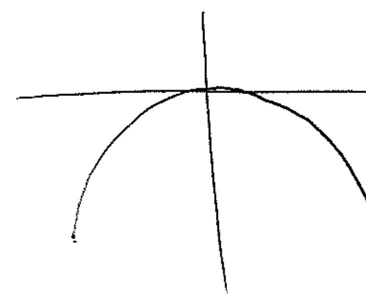
*) $D > 0$ and $f_{xx} > 0$

$\longrightarrow$ (a,b) is a relative minimum



*) $D > 0$ and $f_{xx} < 0$

$\longrightarrow$ (a,b) is a relative maximum



*) $D < 0$

$\longrightarrow$ (a,b) is a saddle point.

*) $D = 0$ $\longrightarrow$ it is inconclusive