Partial derivatives

$f(x,y)$

$f_x = \frac{\partial f}{\partial x}$ = derivative w.r.t. x while y is fixed.

$f_y = \frac{\partial f}{\partial y}$

Ex. $f(x,y) = (x^2+3y^2)e^{2x+y}$

$$f_x = \frac{\partial f}{\partial x} = 2x\, e^{2x+y} + (x^2+3y^2)e^{2x+y}\cdot 2$$
$$= (2x^2+6y^2+2x)e^{2x+y}$$

$$f_y = \frac{\partial f}{\partial y} = 6y\, e^{2x+y} + (x^2+3y^2)e^{2x+y} = (x^2+3y^2+6y)e^{2x+y}$$

§17.2 Applications of partial derivatives

Joint cost function of two products

Consider products A and B

$c = f(x,y)$ = cost of producing x quantity of A and y quantity of B

The marginal cost of product A at production level $x=a$ and $y=b$
(= the cost of producing one more unit of product A)

$$= \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(a,b)}$$

The marginal cost of product B

$$= \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(a,b)}$$

Ex. Joint cost function is given as

$$C = 20x + 15y + 0.2x^2 + 0.3y^2 - 0.1xy.$$

Find the marginal costs $\frac{\partial C}{\partial x}$ and $\frac{\partial C}{\partial y}$ at $x=20$ and $y=10$

$$\frac{\partial C}{\partial x} = \left.(20 + 0.4x - 0.1y)\right|_{(20,10)} = 20 + 0.4\times 20 - 0.1\times 10 = 27.$$

$$\frac{\partial C}{\partial y} = \left.(15 + 0.6y - 0.1x)\right|_{(20,10)} = 15 + 0.6\times 10 - 0.1\times 20$$

$$= 15 + 6 - 2 = 19$$

Production function :

Consider the output of a product at a factory

Let l = amount of labor

k = amount of capital put into the production

$P = f(l, k)$ = the output of the product at l and k.

The marginal productivity w.r.t. l

$$= \frac{\partial P}{\partial l}$$

The marginal productivity w.r.t k

$$= \frac{\partial P}{\partial k}$$

Ex. Production function is given as $P = 10lk - l^2 + 2k^2$

Find the marginal productivities $\frac{\partial P}{\partial l}$ and $\frac{\partial P}{\partial k}$ at $l = 10$ and $k = 5$

$$\frac{\partial P}{\partial l} = \left(10k - 2l\right)\Big|_{(l,k)=(10,5)} = 10 \times 5 - 2 \times 10 = 30$$

$$\frac{\partial P}{\partial k} = (10l + 4k)\Big|_{(l,k)=(10,5)} = 10\times 10 + 4\times 5 = 120$$

Competitive and complementary products.

Consider products A and B

let q_A = quantity demanded for A

p_A = price of A

p_B = price of B

Demand function for A

$$q_A = f(p_A, p_B)$$

q_A is affected by both p_A and p_B.

Ex. Coke and Pepsi

UPS and Fedex

Ex. Airlines and Hotels

Gasoline and tires

$$\frac{\partial q_A}{\partial P_A} < 0 \quad \text{This is always true.} \quad P_A \nearrow \implies q_A \searrow$$

If $\dfrac{\partial q_A}{\partial P_B} > 0$, then we say A and B are <u>competitive</u>

$$P_B \nearrow \implies q_B \searrow \quad q_A \nearrow$$

If $\dfrac{\partial q_A}{\partial P_B} < 0$, then we say A and B are <u>complementary</u>.

$$P_B \nearrow \implies q_B \searrow \quad q_A \searrow$$

<u>Elasticity</u>: Recall the price elasticity of demand for an isolated product

$$PED = \frac{dq}{dp} \cdot \frac{p}{q} < 0$$

Now consider products A and B.

The partial elasticity of demand for A w.r.t. P_A

$$\eta_{P_A} = \frac{\partial q_A}{\partial P_A} \cdot \frac{P_A}{q_A} < 0$$

The partial elasticity of demand for A w.r.t. P_B

$$\eta_{P_B} = \frac{\partial q_A}{\partial P_B} \cdot \frac{P_B}{q_A}$$ may be positive or negative

Ex. $q_A = 60 - 2P_A + 0.5P_B$, $P_A = 10$, $P_B = 20$

Calculate η_{P_A} and η_{P_B}

$$q_A\Big|_{(10,\,20)} = 60 - 2\times10 + 0.5\times20 = 60 - 20 + 10 = \underline{50}$$

$$\eta_{P_A} = \frac{\partial q_A}{\partial P_A} \cdot \frac{P_A}{q_A} = (-2)\times\frac{10}{50} = -0.4$$

$P_A \nearrow 1\%$ $\quad q_A \searrow 0.4\%$

$$\eta_{P_B} = \frac{\partial q_A}{\partial P_B} \cdot \frac{P_B}{q_A} = (0.5)\times\frac{20}{50} = 0.2$$

$P_B \nearrow 1\%$ $\quad q_A \nearrow 0.2\%$. competitive.

Taylor polynomials for functions of several variables.

(supplementary note #2)

The one-variable case: $f(t)$

Background: Find minimum and maximum values of $f(t)$

Step 1: Solve $f'(t)=0$ for critical points.

Step 2: Study the behavior of $f(t)$ near each critical point

The second order approximation near t_0.

$$f(t) \approx T_2(t) = C_0 + C_1(t-t_0) + C_2(t-t_0)^2$$

Set $f(t_0)=T_2(t_0) \longrightarrow C_0=f(t_0)$

Set $f'(t_0)=T_2'(t_0) \longrightarrow C_1=f'(t_0)$

Set $f''(t_0)=T_2''(t_0) \longrightarrow 2C_2=f''(t_0) \longrightarrow C_2=\frac{1}{2}f''(t_0)$

$$\boxed{T_2(t) = f(t_0) + f'(t_0)(t-t_0) + \frac{1}{2} f''(t_0)(t-t_0)^2}$$

This is the second order approximation.

$$\boxed{T_1(t) = f(t_0) + f'(t_0)(t-t_0)}$$

First order approximation.

The two-variable case $F(x, y)$

The first order approximation near (x_0, y_0)

$$F(x,y) \approx T_1(x,y) = C_{0,0} + C_{1,0}(x-x_0) + C_{0,1}(y-y_0)$$

Set $F(x_0, y_0) = T_1(x_0, y_0) \longrightarrow C_{0,0} = F(x_0, y_0)$

Set $F_x(x_0, y_0) = T_{1x}(x_0, y_0) \longrightarrow C_{1,0} = F_x(x_0, y_0)$

Set $F_y(x_0, y_0) = T_{1y}(x_0, y_0) \longrightarrow C_{0,1} = F_y(x_0, y_0)$

$$\boxed{T_1(x,y) = F(x_0, y_0) + F_x(x_0, y_0)(x-x_0) + F_y(x_0, y_0)(y-y_0)}$$

§ 17.4 Higher order partial derivatives.

$f(x, y)$

$\dfrac{\partial f}{\partial x}$ is a function of (x, y)

$\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$ = derivative of $\dfrac{\partial f}{\partial x}$ w.r.t. x while y is fixed.

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$ = derivative of $\dfrac{\partial f}{\partial y}$ w.r.t y while x is fixed.

$\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ = derivative of $\dfrac{\partial f}{\partial x}$ w.r.t. y while x is fixed.

<u>Notation</u>:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = (f_x)_x = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = (f_y)_y = f_{yy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}.$$

<u>Caution</u>: f_{xy} and f_{yx} have <u>different meanings</u>.

<u>Good news</u> For all functions in real applications,

$$f_{xy} = f_{yx}$$

<u>Ex.</u> $F(x,y) = e^{x^2+2y^2}$

$$F_x = e^{x^2+2y^2}(2x)$$

$$F_{xx} = (F_x)_x = e^{x^2+2y^2}(2x)^2 + e^{x^2+2y^2} \cdot 2$$

$$= e^{x^2+2y^2}(4x^2+2)$$

$$F_{xy} = (F_x)_y = e^{x^2+2y^2}(4y)(2x) = \underline{e^{x^2+2y^2} 8xy}$$

$$F_y = e^{x^2+2y^2}(4y)$$

$$F_{yy} = (F_y)_y = e^{x^2+2y^2}(4y)^2 + e^{x^2+2y^2} \cdot 4$$

$$= e^{x^2+2y^2}(16y^2+4)$$

$$F_{yx} = (F_y)_x = e^{x^2+2y^2}(2x)(4y) = \underline{e^{x^2+2y^2} 8xy}$$