Recap

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

Integration by partial fractions

Recall $\int \frac{1}{u+a}\,du = \ln|u+a| + C$

Ex $\int \frac{x+2}{x^2+8x+15}\,dx$

$$\frac{x+2}{(x+5)(x+3)} = \frac{A}{(x+5)} + \frac{B}{(x+3)} = \frac{A(x+3)+B(x+5)}{(x+5)(x+3)}$$

$\longrightarrow \quad x+2 = A(x+3) + B(x+5)$

At $x=-5 \rightarrow -3 = A(-2) \rightarrow 3 = 2A \rightarrow A = \frac{3}{2}$

At $x=-3 \rightarrow -1 = 2B \rightarrow B = -\frac{1}{2}$

$$\int \frac{x+2}{x^2+8x+15}\,dx = \int \left(\frac{3}{2}\frac{1}{(x+5)} - \frac{1}{2}\frac{1}{(x+3)}\right) dx$$

$$= \frac{3}{2}\ln|x+5| - \frac{1}{2}\ln|x+3| + C$$

Integration by tables

Ex $\int \sqrt{x^2+9}\,dx$, $\int x^3(\ln x)^2\,dx$

Average value of a function

$$\bar{f} = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$

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§15.5 Differential equations (DE)

An equation involving derivative(s) of unknown function $y(x)$.

Ex $y' = 2x^2y^2$

This is a first order differential equation (DE)

A solution of $\underset{\text{(DE)}}{y' = 2x^2y^2}$ is a function $y(x)$ that satisfies $\underset{\text{(DE)}}{y' = 2x^2y^2}$

Method: Separation of variables

Ex. $y' = 2x^2 y^2$

$\rightarrow \dfrac{dy}{dx} = 2x^2 y^2$

$\rightarrow \int \frac{1}{y^2} dy = \int 2x^2 dx$

$$\frac{y^{-2+1}}{-2+1} = 2\frac{x^{2+1}}{2+1} + C$$

$$-\frac{1}{y} = \frac{2}{3}x^3 + C$$

$$y = \frac{-1}{\frac{2}{3}x^3 + C}$$

Verify it

$$y' = \frac{1}{(\frac{2}{3}x^3+C)^2} \cdot \frac{d}{dx}\left(\frac{2}{3}x^3 + C\right)$$
$$= y^2 2x^2$$

$$\frac{d}{du}\left(\frac{-1}{u}\right) = \frac{1}{u^2}$$

This satisfies DE for any C

$y = \dfrac{-1}{\frac{2}{3}x^3 + C}$ is a family of solutions of $y' = 2x^2y^2$

the general solution of DE

With an additional condition, we can determine a unique $y(x)$

Ex $\begin{cases} y' = 2x^2y^2 \\ y(1) = -1 \end{cases}$

$$y' = 2x^2y^2 \longrightarrow \quad y = \frac{-1}{\frac{2}{3}x^3 + C}$$

$$y(1) = -1 \longrightarrow \quad -1 = \frac{-1}{\frac{2}{3}\times 1^3 + C} \longrightarrow \quad 1 = \frac{1}{\frac{2}{3} + C}$$

$$\longrightarrow \frac{2}{3} + C = 1 \longrightarrow \quad C = \frac{1}{3}$$

$$\boxed{y(x) = \frac{-1}{\frac{2}{3}x^3 + \frac{1}{3}}}$$

Ex. $y' = x^2 \ln(x^3+2)$

$$\frac{dy}{dx} = x^2 \ln(x^3+2)$$

$$\rightarrow \quad \int dy = \int x^2 \ln(x^3+2)\, dx$$

LHS $= y$

RHS $= \int \underbrace{\frac{1}{3} \ln(x^3+2)}_{\frac{1}{3}\ln u} \underbrace{3x^2 dx}_{du}$

Try $u = x^3+2$

$du = 3x^2 dx$

$x^2 = \frac{1}{3} \cdot (3x^2)$

$= \frac{1}{3} \int \ln u \, du$

$$\boxed{\text{Item 41}: \int \ln u \, du = u \ln u - u + C}$$

$= \frac{1}{3}(u \ln u - u) + C$

LHS = RHS $\rightarrow$ $\boxed{y = \frac{1}{3}(x^3+2)\ln(x^3+2) - \frac{1}{3}(x^3+2) + C}$

Ex $y' = yx$

$$\frac{dy}{dx} = yx$$

$$\rightarrow \int \frac{1}{y}\,dy = \int x\,dx$$

$$\rightarrow \ln|y| = \frac{1}{2}x^2 + C_1$$

$$\rightarrow |y| = e^{\frac{1}{2}x^2 + C_1} = e^{\frac{1}{2}x^2} \cdot e^{C_1}$$

$$\rightarrow y = \underbrace{\pm e^{C_1}}_{C} e^{\frac{1}{2}x^2}$$

$$y = C e^{\frac{1}{2}x^2}$$

C may be positive or negative

Sometimes the general solution $y(x)$ is given implicitly in an equation.

Ex. $y' = \dfrac{x^3+1}{y^2+1}$

$$\frac{dy}{dx} = \frac{x^3+1}{y^2+1}$$

$$\int (y^2+1)\,dy = \int (x^3+1)\,dx$$

$$\frac{y^3}{3} + y = \frac{x^4}{4} + x + C$$

$$\rightarrow \boxed{\frac{y^3}{3} + y - \frac{x^4}{4} - x = C}$$

Exponential growth and decay.

Consider $\dfrac{dy}{dt} = r y$,

r = constant independent of t and y

$$\int \frac{1}{y}\, dy = \int r\, dt$$

$$\ln|y| = rt + c_1$$

$$|y| = e^{c_1} e^{rt}$$

$$y = \pm e^{c_1} e^{rt}$$

In many applications, $y > 0$.

$$y(t) = c\, e^{rt}, \qquad c > 0$$

At $t=0$. $\quad y(0) = c \longrightarrow \boxed{c = y(0)}$

$$\longrightarrow \boxed{y(t) = y(0)\, e^{rt}}$$

$\underline{r>0}$ exponential growth, r is called growth rate

$\underline{r<0}$ exponential decay, $\lambda = (-r)$ is called decay rate.

Ex. US population growth.

$$\left.\begin{array}{l} 1980:\ 227 \text{ millions} \\ 2000:\ 282 \text{ millions} \end{array}\right\} \text{ two data points}$$

Suppose US population grows at a constant rate

Find $y(t)$, population at t years past 1980.

$y(0) = 227$
$y(20) = 282$ two data points.

$$y(t) = \underbrace{y(0)}_{227} e^{rt}$$

$\boxed{y(20)=282} \longrightarrow 282 = 227 e^{r \times 20}$

$\longrightarrow e^{20r} = (282/227)$

$\longrightarrow 20r = \ln(282/227)$

$\longrightarrow r = \frac{1}{20}\ln(282/227) \approx 0.0108 = 1.08\%/\text{year}.$

$$y(t) = 227\, e^{0.0108 t}$$

Predicted population at 2010

$$y(30) = 227\, e^{0.0108 \times 30} \approx 314$$

Real data: US population at 2010 $= 309$ millions.

Exponential decay and half life

$$y(t) = y(0)\, e^{-\lambda t} \qquad \lambda > 0 \quad \text{decay rate}$$

half ~~time~~ life, $t_h =$ time for y to decay by half

$$y(0)\, e^{-\lambda t_h} = 0.5\, y(0)$$

$$\longrightarrow \quad e^{-\lambda t_h} = 0.5$$

$$\longrightarrow \quad -\lambda t_h = -\ln(2)$$

$$\longrightarrow \quad \boxed{t_h = \frac{1}{\lambda} \ln(2)}$$

Ex ^{14}C is radioactive and decays with a half life of 5730 years.

Find the decay rate λ

$$t_h = \frac{1}{\lambda} \ln(2)$$

$$\longrightarrow \quad \lambda = \frac{1}{t_h} \ln(2) = \frac{1}{5730} \ln(2) \approx 0.00012097$$

How to estimate the age of an ancient tool.

Consider ^{14}C to ^{12}C ratio.

In the atomsphere

radioactive decay of ^{14}C $\searrow$ + production of ^{14}C by cosmic rays $\nearrow$

$\longrightarrow$ ^{14}C to ^{12}C ratio $= 10^{-12}$, constant, invariant with t.

Inside an object, after is stops absorbing C from the atomsphere ^{14}C to ^{12}C ratio decays with a half life of 5730 years.

$$y(t) = y(0)\, e^{-\lambda t} \qquad \lambda = \frac{\ln(2)}{5730}$$

Ex. A scroll was found to has a ^{14}C to ^{12}C ratio 80% of the corresponding ratio in present day similar material.

Estimate the age of scroll

$$\underbrace{y(t)}_{80\%\, y(0)} = y(0)\, e^{-\lambda t}$$

$\longrightarrow \quad 0.8 = e^{-\lambda t}$

$\longrightarrow \quad -\lambda t = \ln(0.8)$

$\longrightarrow \quad t = -\frac{1}{\lambda} \ln(0.8)$

$\longrightarrow \quad t = -\frac{\ln(0.8)}{\ln(2)} 5730 \approx 1845$