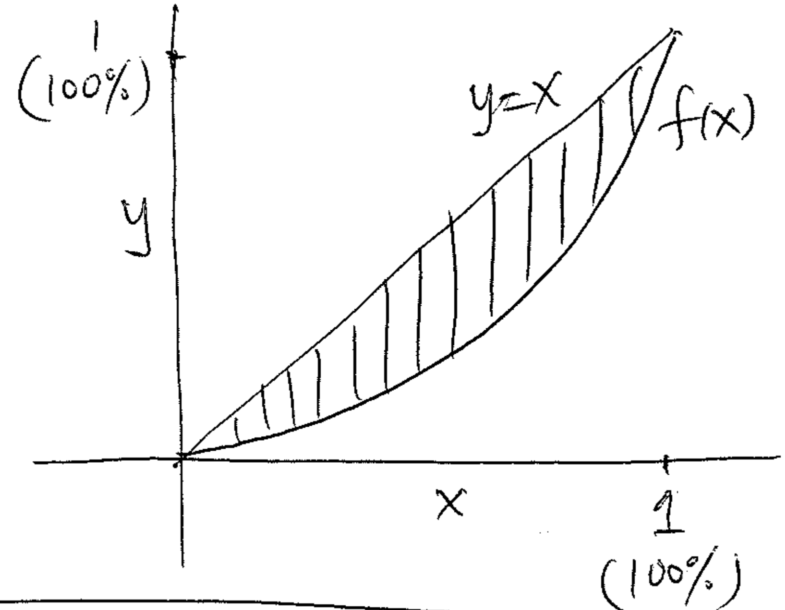
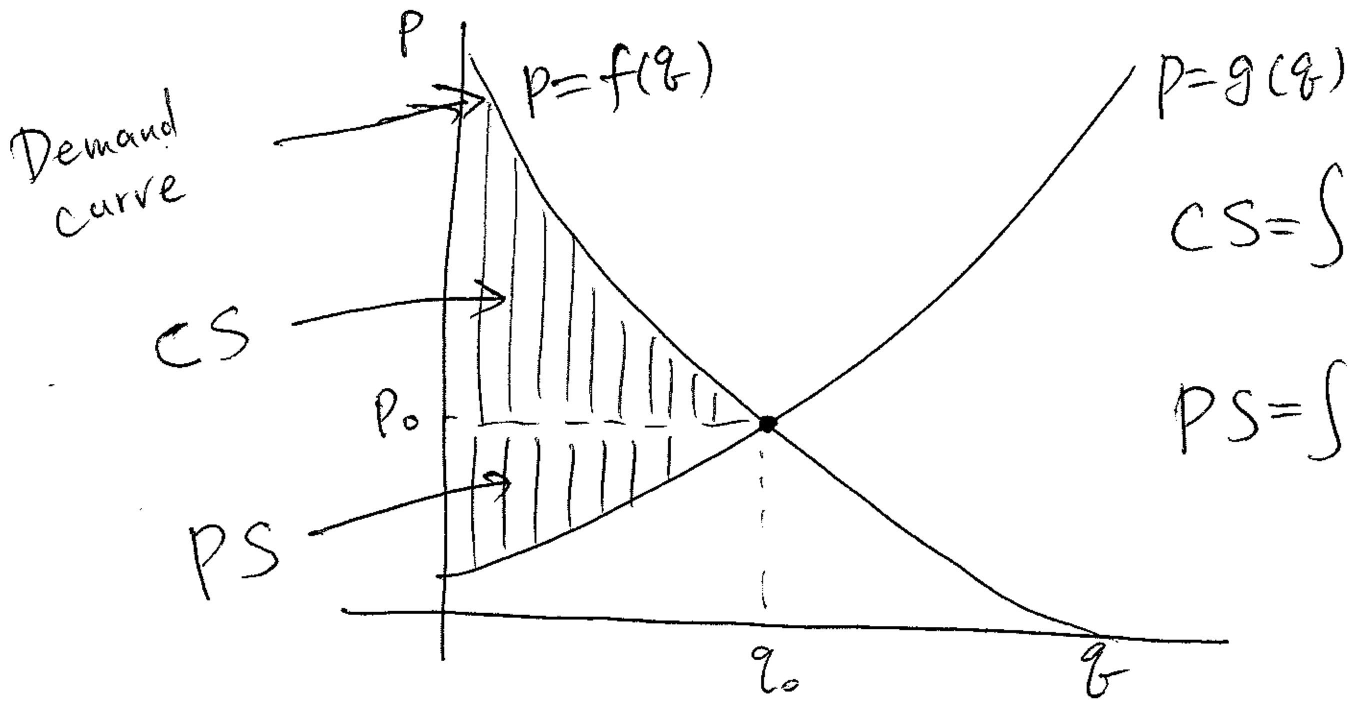
Recap Lorenz curve for income distribution

coefficient of inequality (Gini coefficient)

$$G = \frac{\int_0^1 (x - f(x))\,dx}{\frac{1}{2}}$$



Consumers' surplus (CS) and producers' surplus (PS)



$$CS = \int_0^{q_0} (f(q) - p_0)\,dq$$

$$PS = \int_0^{q_0} (p_0 - g(q))\,dq$$

§15.1 Integration by parts

product rule (for derivatives)

$$(u(x)v(x))' = u(x)v'(x) + u'(x)v(x)$$

$$\rightarrow \underline{\int u(x)v'(x)dx} + \int v(x)u'(x)dx = \int (u(x)v(x))'dx$$

$$\rightarrow \int u(x)\underbrace{v'(x)dx}_{dv} = u(x)v(x) - \int v(x)\underbrace{u'(x)dx}_{du}$$

A concise form

$$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$

Formula for integration by parts

<u>Ex</u>. $\int \underset{u(x)}{x}\, \underset{v'(x)}{e^x}\, dx$. Try $u=x$, $v'(x)=e^x \to v(x)=e^x$

$$= \int x\, d(e^x) = x e^x - \int e^x dx \boxed{= x e^x - e^x + C}$$

<u>Ex</u>. $\int \sqrt{x} \ln x \, dx$ Try $u=\sqrt{x}$, $v'(x)=\ln x$, $v(x)=\,?$

$$= \int \ln x \, d\left(\tfrac{2}{3} x^{\frac{3}{2}}\right)$$ Try $u=\ln x$, $v'(x)=\sqrt{x}$,

$$v(x)=\int \sqrt{x}\, dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3} x^{\frac{3}{2}}$$

$$= \ln x \left(\tfrac{2}{3} x^{\frac{3}{2}}\right) - \int \tfrac{2}{3} x^{\frac{3}{2}} \tfrac{1}{x}\, dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \int x^{\frac{1}{2}} dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{2}{3} \cdot \frac{2}{3} x^{\frac{3}{2}} + C \boxed{= \frac{2}{3} x^{\frac{3}{2}} \ln x - \frac{4}{9} x^{\frac{3}{2}} + C}$$

General guideline of selecting $u(x)$, $v'(x)$

$u(x)$ = something we like to get rid of when calculating $u'(x)$

$v'(x)$ = something we can integrate.

Ex $\int \ln x \, dx$ Try $u(x) = \ln x$, $v'(x) = 1 \to v(x) = x$

$= (\ln x)x - \int x \, d(\ln x)$

$= x(\ln x) - \int 1 \, dx \boxed{= x \ln x - x + C}$

Ex. $\int x^3 e^{x^2} dx$ First do substitution

$= \int \underbrace{\frac{1}{2} x^2 e^{x^2}}_{\frac{1}{2} w e^{w}} \underbrace{2x \, dx}_{dw}$ Try $w = x^2$, $dw = 2x \, dx$

$x^3 = \frac{1}{2} x^2 \cdot 2x$

$= \frac{1}{2} \int w e^w dw = \frac{1}{2}(w e^w - e^w) + C$

$\boxed{= \frac{1}{2}(x^2 e^{x^2} - e^{x^2}) + C}$

Integration by parts for the definite integral

$$\int_a^b u \, dv = (uv)\Big|_a^b - \int_a^b v \, du$$

§15.2 Partial Fractions

Ex $\displaystyle\int \frac{x^3}{x^2+3x+2}\,dx$

First, we reduce it to a proper rational function.

$$
\begin{array}{r|l}
 & \quad x - 3 \\
\hline
x^2+3x+2 & x^3 \\
 & x^3+3x^2+2x \\
\hline
 & \quad -3x^2-2x \\
 & \quad -3x^2-9x-6 \\
\hline
 & \qquad 7x+6
\end{array}
$$

$$\frac{x^3}{x^2+3x+2} = (x-3) + \frac{7x+6}{x^2+3x+2}$$

we focus on this part

Factor the denominator. $(x+2)(x+1)$

We write $\dfrac{7x+6}{(x+2)(x+1)}$ as a sum of partial fractions.

$$\frac{7x+6}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

Recall $\int \frac{1}{x+a}\,dx = \ln|x+a| + C$

$$RHS = \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$$

$$\longrightarrow \boxed{7x+6 = A(x+1) + B(x+2)}$$

At $x=-2$, $\quad -8 = -A \longrightarrow A = 8$

At $x=-1$, $\quad -1 = B \longrightarrow B = -1$

$$\int \frac{x^3}{x^2+3x+2}\,dx = \int \left((x-3) + \frac{8}{x+2} - \frac{1}{x+1} \right) dx$$

$$= \frac{x^2}{2} - 3x + 8\ln|x+2| - \ln|x+1| + C$$

Ex $\int \frac{1}{x^2-7x+6}\,dx$

Factor the denominator

$$x^2-7x+6 = (x-6)(x-1)$$

Write it as a sum of partial fractions

$$\frac{1}{(x-6)(x-1)} = \frac{A}{x-6} + \frac{B}{x-1} = \frac{A(x-1)+B(x-6)}{(x-6)(x-1)}$$

$$\longrightarrow \quad 1 = A(x-1)+B(x-6)$$

At $x=6$, $1 = 5A \longrightarrow A=\frac{1}{5}$

At $x=1$, $1 = -5B \longrightarrow B=\frac{-1}{5}$

$$\int \frac{1}{x^2-7x+6}\,dx = \int \left(\frac{1}{5}\frac{1}{(x-6)} - \frac{1}{5}\frac{1}{(x-1)}\right) dx$$

$$= \frac{1}{5}\ln|x-6| - \frac{1}{5}\ln|x-1| + C$$

§15.3 Integration by tables

Ex $\int \sqrt{x^2+4}\,dx$

Item 23 $\int \sqrt{u^2 \pm a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln\left|u+\sqrt{u^2 \pm a^2}\right|\right) + C$

$$\int \sqrt{x^2+4}\,dx = \frac{1}{2}\left(x\sqrt{x^2+4} + 4\ln\left|x+\sqrt{x^2+4}\right|\right) + C$$

Ex $\int x(\ln x)^2\,dx$

$\ln^m u = (\ln u)^m$

Item 43 $\int u^n (\ln u)^m\,du = \frac{u^{n+1}}{n+1}(\ln u)^m - \frac{m}{n+1}\int u^n (\ln u)^{m-1}\,du$

$n \neq -1$

$\int x(\ln x)^2\,dx$, $\quad n=1,\ m=2$

$$= \frac{x^2}{2}(\ln x)^2 - \frac{2}{2}\underbrace{\int x(\ln x)\,dx} \qquad n=1,\ m=1$$

$$= \frac{x^2}{2}(\ln x)^2 - \left[\frac{x^2}{2}(\ln x) - \frac{1}{2}\int x\,dx\right]$$

$$= \frac{x^2}{2}(\ln x)^2 - \frac{x^2}{2}(\ln x) + \frac{1}{4}x^2 + C$$

Integration applied to annuity.

<u>An annuity</u> is a series of payments made at equal intervals over time period $[0, T]$

<u>Ex</u>. monthly payments of \$100 over next 10 years

<u>A continuous annuity</u>:

The payment is continuous in time with <u>a give rate</u>.

<u>Ex</u>. Continuous payment at the rate of \$1200 per year.

Let $f(t)$ be the rate of payment at time t.

payment in $[t, t+dt] = f(t)\,dt$, $0 \leq t \leq T$

Its value at $t=0$ (now) $= e^{-rt} f(t)\,dt$	It value at $t=T$ (end of annuity) $= e^{r(T-t)} f(t)\,dt$

The present value of the annuity

$$A = \int_0^T f(t) e^{-rt} dt$$

The accumulated amount of the annuity

$$S = \int_0^T f(t) e^{r(T-t)} dt$$

Ex. $f(t) = 1200$, $T = 10$, $r = 0.05$

$$A = \int_0^{10} 1200 e^{-0.05t} dt$$

$$\boxed{\text{Recall } \int e^{ax} dx = \frac{1}{a} e^{ax} + C}$$

$$= \frac{1200}{(-0.05)} e^{-0.05t} \Big|_0^{10} = -24000 (e^{-0.5} - 1)$$

$$= 24000 (1 - e^{-0.5}) \approx 9{,}443.26$$

$$e^{0.5} \cdot e^{-0.05t}$$

$$S = \int_0^{10} 1200 e^{0.05(10-t)} dt = \int_0^{10} 1200 e^{0.5 - 0.05t} dt$$

$$= 1200 e^{0.5} \int_0^{10} e^{-0.05t} dt = 24000 e^{0.5} (1 - e^{-0.5})$$

$$\approx 15{,}569.31$$

§15.4 Average value of a function.

$$\bar{f} = \frac{1}{(b-a)}\int_a^b f(x)\,dx$$

Average value of f(x) over [a, b]

EX. $F(t) = \dfrac{F_0}{\left(1+\frac{t^2}{4}\right)^{3/2}}$ Blood flow

Find the average flow $\bar{F}$ over [0, T]

$$F(t) = \frac{F_0}{\left(\frac{1}{4}\right)^{3/2}(4+t^2)^{3/2}} = \frac{8F_0}{(t^2+4)^{3/2}}$$

$\boxed{\text{Item 32}}$ $\displaystyle\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{\pm u}{a^2\sqrt{(u^2 \pm a^2)}} + C$

$$\int_0^T F(t)\,dt = 8F_0 \cdot \int \frac{1}{(t^2+4)^{3/2}}\,dt$$

$$= 8F_0 \left.\frac{t}{4\sqrt{t^2+4}}\right|_0^T = 2F_0 \left.\frac{t}{\sqrt{t^2+4}}\right|_0^T$$

$$= 2F_0 \frac{T}{\sqrt{T^2+4}} = F_0 \frac{T}{\sqrt{\frac{T^2}{4}+1}}$$

$$\overline{F} = \frac{1}{T}\int_0^T F(t)\,dt = \frac{F_0}{\sqrt{\frac{T^2}{4}+1}}$$