Recap

Fundamental theorem of calculus

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$$

properties of the definite integral

For $b<a$, $\int_a^b f(x)\,dx \overset{\text{defined as}}{=\!=} -\int_b^a f(x)\,dx$

Ex $\int_3^1 f(x)\,dx = -\int_1^3 f(x)\,dx$

Area between curves

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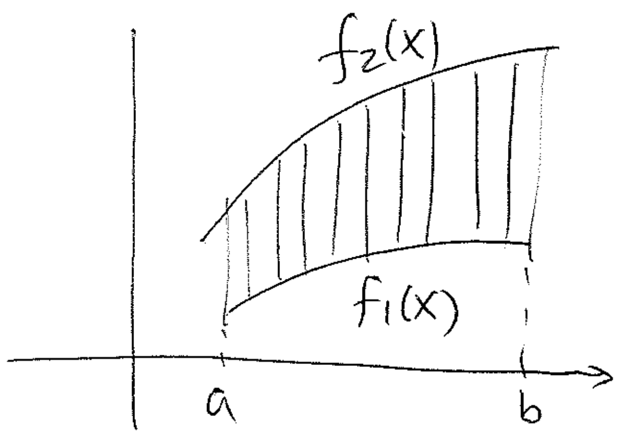
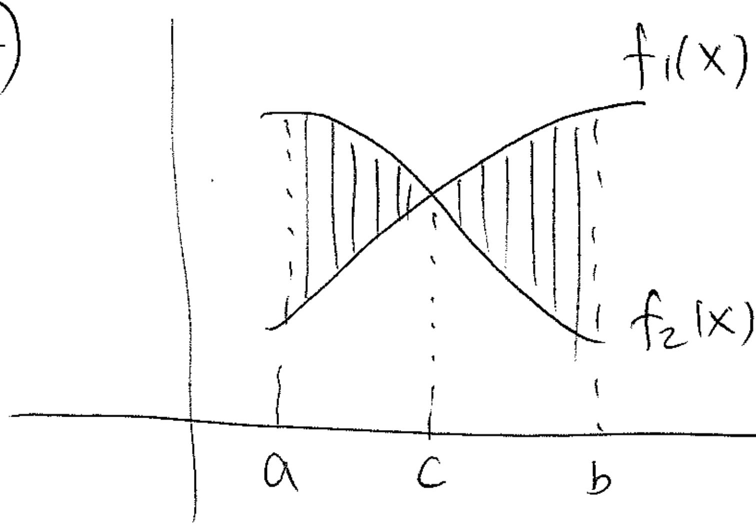


$$\text{Area} = \int_a^b \left(f_2(x) - f_1(x)\right) dx$$

a and b are given

*)



a and b are given

Find c

$$\text{Area} = \int_a^c (f_2(x) - f_1(x))\,dx + \int_c^b -(f_2(x) - f_1(x))\,dx$$

Only need to find

$$\int (f_2(x) - f_1(x))\,dx$$

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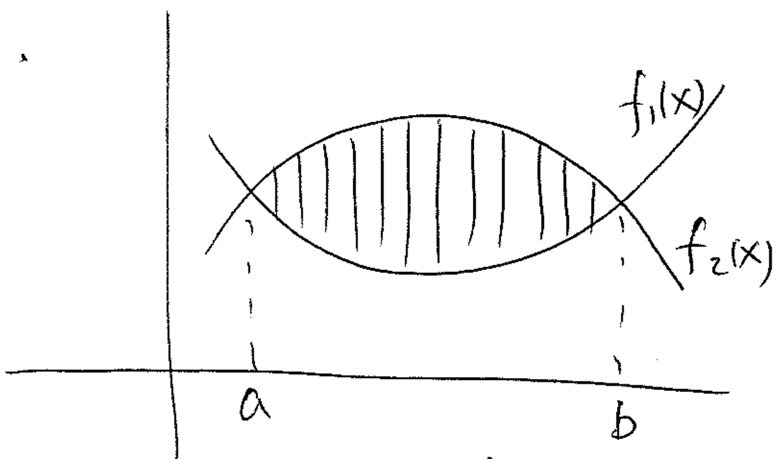


a and b are not given

Find a and b by solving

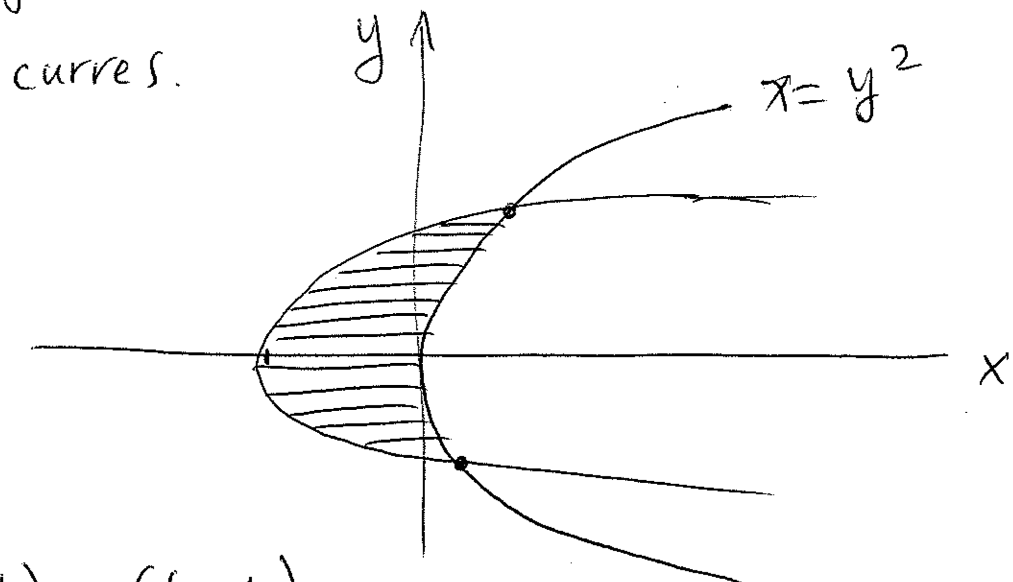
$$f_2(x) = f_1(x)$$

$$\text{Area} = \int_a^b (f_2(x) - f_1(x))\,dx$$

Horizontal Strips

Ex. The area bounded by $x = y^2$ and $x = 2y^2 - 1$

We first
Find the intersections of two curves.



$$2y^2 - 1 = y^2$$

$$\rightarrow \quad y^2 = 1$$

$$\rightarrow \quad y = \pm 1$$

Two intersections: $(1, -1)$, $(1, 1)$

$$\text{Area} = \int \begin{pmatrix} \text{horizontal width} \\ \text{at level } y \end{pmatrix} dy$$

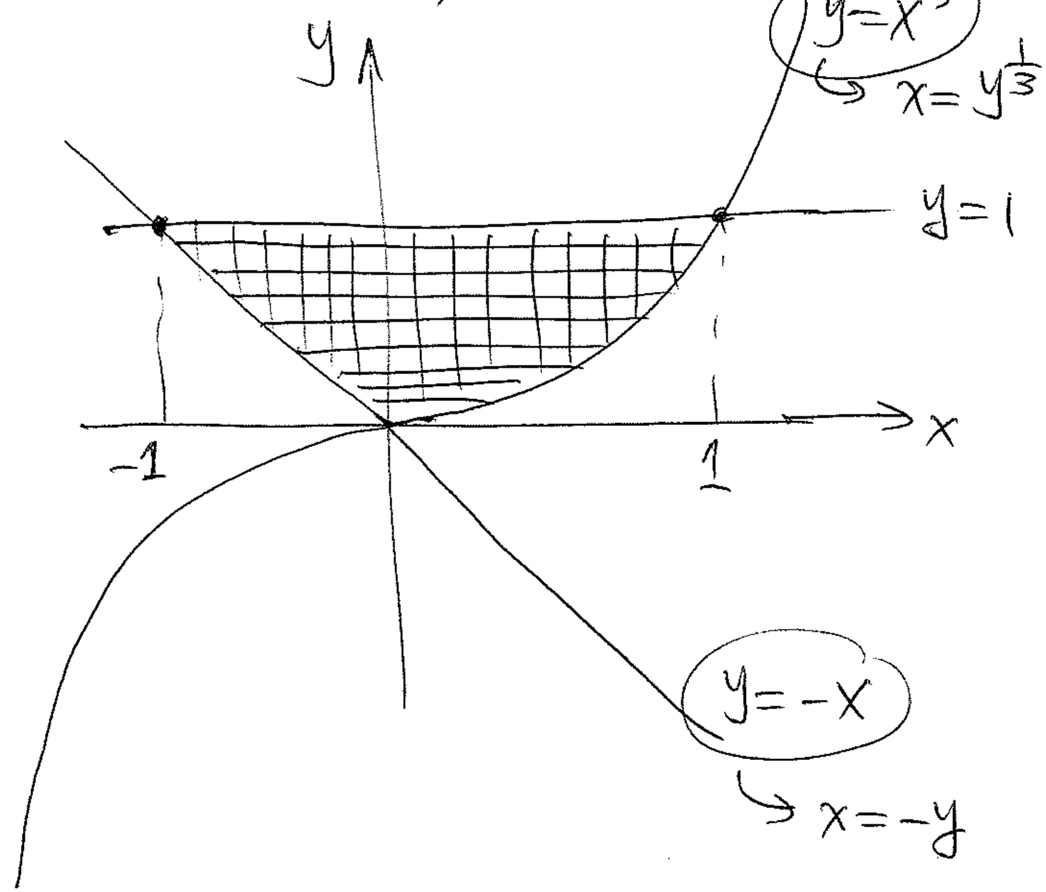
$$= \int_{-1}^{1} \left(y^2 - (2y^2 - 1) \right) dy = \int_{-1}^{1} (1 - y^2)\, dy$$

$$\int (1 - y^2)\, dy = y - \frac{1}{3} y^3 + C$$

$$\text{Area} = \left(y - \frac{y^3}{3}\right)\Big|_{-1}^{1} = \left((1-(-1)) - \frac{1}{3}(1-(-1))\right)$$

$$= 2 - \frac{2}{3} = \boxed{\frac{4}{3}}$$

Ex. The area bounded by $y = x^3$, $y = -x$, and $y = 1$



Horizontal Strips

$$\text{Area} = \int_0^1 (\text{horizontal width})\,dy$$

$$= \int_0^1 \left(y^{\frac{1}{3}} - (-y)\right) dy$$

$$= \int_0^1 \left(y^{\frac{1}{3}} + y\right) dy$$

$$= \left(\frac{y^{\frac{1}{3}+1}}{\frac{1}{3}+1} + \frac{1}{2}y^2\right)\Big|_0^1$$

$$= \left(\frac{3}{4}y^{4/3} + \frac{1}{2}y^2\right)\Big|_0^1$$

$$= \frac{3}{4} + \frac{1}{2} = \boxed{\frac{5}{4}}$$

$\boxed{\text{Vertical strips}}$

$$\text{Area} = \int_{-1}^{1} (\text{vertical height})\, dx$$

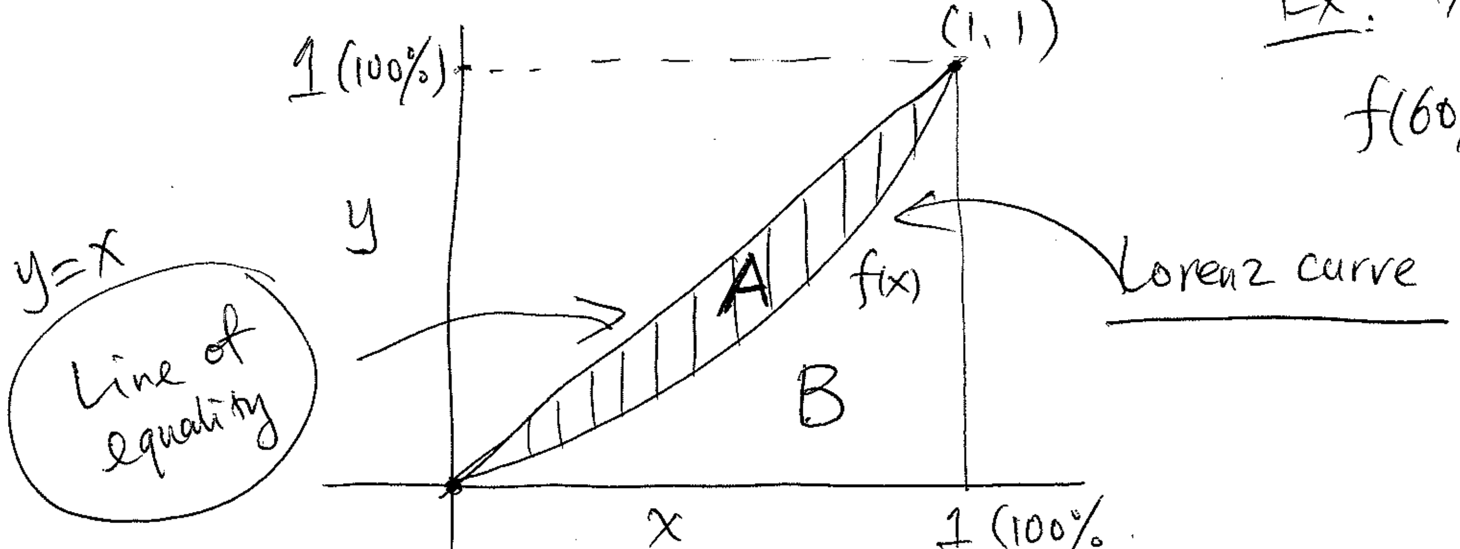
$$= \int_{-1}^{0} (1-(-x))\,dx + \int_{0}^{1} (1-x^3)\,dx$$

$$= \cdots$$

Lorenz curve for income distribution.

Ex. $x = 60\%$

$f(60\%) = 26.6\%$



x : cumulative share of population from lowest to highest incomes

y : cumulative share of income

Coefficient of inequality $= \dfrac{A}{A+B}$

also called the Gini coefficient

$$A+B = \frac{1}{2}$$

$$A = \int_0^1 (x - f(x))\, dx$$

$$\boxed{G = \frac{\int_0^1 (x - f(x))\, dx}{\frac{1}{2}}}$$

Ex. Given the Lorenz curve $y = \frac{2}{3}x^3 + \frac{1}{3}x$. Find the Gini coefficient G.

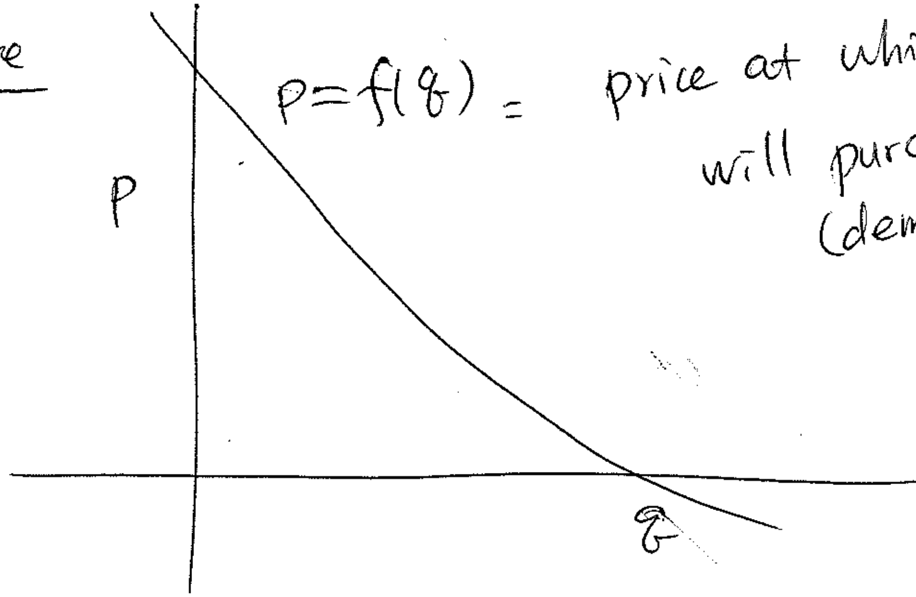
$$A = \int_0^1 \left(x - \left(\frac{2}{3}x^3 + \frac{1}{3}x\right)\right) dx = \int_0^1 \left(\frac{2}{3}x - \frac{2}{3}x^3\right) dx$$

$$= \left(\frac{2}{3}\cdot\frac{x^2}{2} - \frac{2}{3}\cdot\frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{3}(1-0) - \frac{1}{6}(1-0) = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

$$G = \frac{\frac{1}{6}}{\frac{1}{2}} \boxed{= \frac{1}{3}}$$

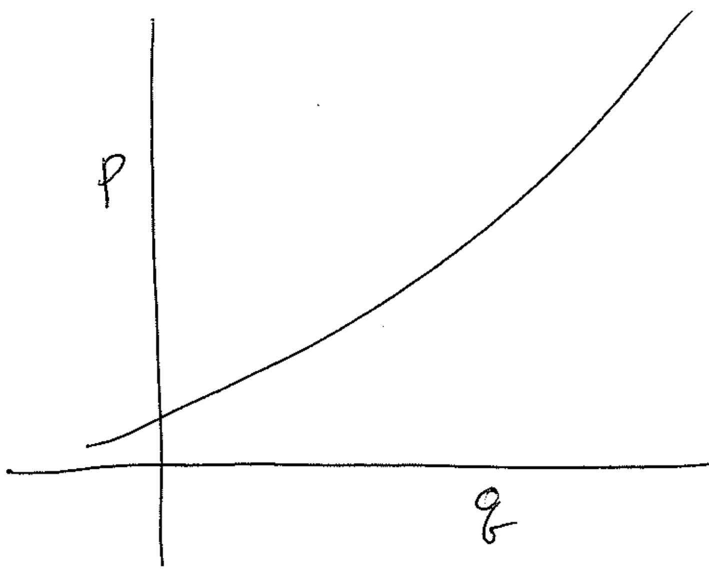
§14.10 Consumers' surplus, producers' surplus

Demand curve



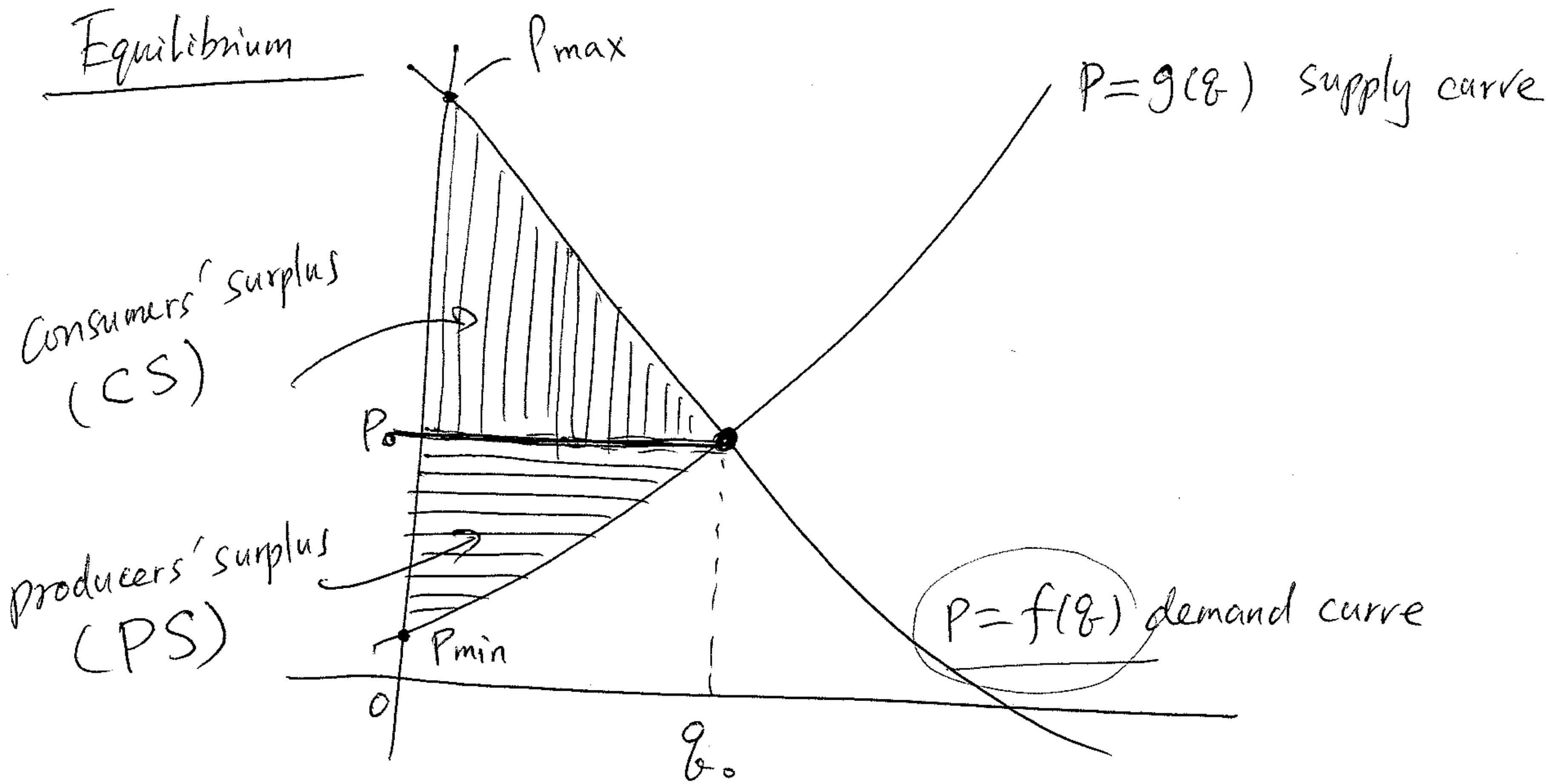
$p = f(q)$ = price at which consumers will purchase quantity q (demand)

Supply curve



$p = g(q)$ = price at which producers will sell quantity q (supply)

Equilibrium



P_0 = price at which quantity demanded by consumers = quantity supplied by producers

P_{max}: maximum price consumers are willing to pay.

P_{min}: minimum price producers are willing to accept

Observation:

*) If the price is fixed at ~~a higher value~~ a value $> P_0$
some consumers are still willing to purchase.

*) If the price is fixed at a value $< P_0$
Some producers are still willing to supply.

$$CS = \int_0^{q_0} (f(q) - P_0)\, dq$$

$$CS = \int_{P_0}^{P_{max}} q_{demand}(p)\, dp.$$

$$PS = \int_0^{q_0} (P_0 - g(q))\, dq$$

$$PS = \int_{P_{min}}^{P_0} q_{supply}(p)\, dp.$$

Ex. Given: demand curve $p = 900 - q^2$ — $f(q)$

supply curve $p = 10q + 300$ — $g(q)$

Find Consumers' surplus (CS) and producers' surplus (PS)

We first find the equilibrium (p_0, q_0)

$$900 - q^2 = 10q + 300$$

$$\longrightarrow 0 = q^2 + 10q - 600$$

$$\longrightarrow 0 = (q+30)(q-20)$$

$$\longrightarrow \boxed{q_0 = 20}$$

$$\longrightarrow \boxed{p_0 = 900 - (20)^2 = 500}$$

$$CS = \int_0^{q_0} (f(q) - p_0)\, dq = \int_0^{20} (900 - q^2 - 500)\, dq$$

$$= \int_0^{20} (400 - q^2)\, dq = \int_0^{20} 400\, dq - \int_0^{20} q^2\, dq$$

$$= 400q\Big|_0^{20} - \frac{1}{3}q^3\Big|_0^{20} = 400 \times 20 - \frac{1}{3}(20)^3 = \left(1 - \frac{1}{3}\right) \times 8000 = \frac{16000}{3}$$

$$PS = \int_0^{q_0} (p_0 - g(q))\,dq = \int_0^{20} \left(500 - (10q + 300)\right) dq$$

$$= \int_0^{20} (200 - 10q)\,dq = \left(200q - 10 \cdot \frac{q^2}{2}\right)\Big|_0^{20}$$

$$= 200 \times 20 - 5 \times (20)^2 = 4000 - 2000 \boxed{= 2000}$$

Ex. Given: demand $q = 100 - p^2$

supply: $p = \frac{q}{25} + 2$

Find CS and PS

We first find the equilibrium (p_0, q_0)

$$p = \frac{100 - p^2}{25} + 2$$

$$\longrightarrow 25p = 100 - p^2 + 50$$

$$\longrightarrow p^2 + 25p - 150 = 0$$

$$\longrightarrow (p+30)(p-5) = 0$$

$$\longrightarrow \boxed{p_0 = 5} \qquad \boxed{q_0 = 100 - p_0^2 = 75}$$

Find P_{max}: $q = 0 \longrightarrow 100 - (P_{max})^2 = 0$

$$\longrightarrow P_{max} = 10$$

$$CS = \int_{P_0}^{P_{max}} q_{demand}(P)\, dP$$

$$= \int_5^{10} (100 - P^2)\, dP$$

$$= \left(100P - \frac{1}{3}P^3\right)\Big|_5^{10}$$

$$= 100 \times (10-5) - \frac{1}{3}(10^3 - 5^3)$$

$$= 500 - \frac{1}{3}(1000 - 125) = \frac{625}{3}$$