Recap

Summation notation

$$\sum_{k=m}^{n} A_k = \underbrace{A_m}_{k=m} + \underbrace{A_{m+1}}_{k=m+1} + \cdots + \underbrace{A_n}_{k=n} \qquad n \geq m$$

Summation Formulas

$$S_{n,0} = \sum_{k=1}^{n} 1 = n \ , \qquad \lim_{n\to\infty} \frac{1}{n} S_{n,0} = 1$$

$$S_{n,1} = \sum_{k=1}^{n} k = \frac{n^2+n}{2} \ , \qquad \lim_{n\to\infty} \frac{1}{n^2} S_{n,1} = \frac{1}{2}$$

$$S_{n,2} = \sum_{k=1}^{n} k^2 = \frac{2n^3+3n^2+n}{6} \ , \qquad \lim_{n\to\infty} \frac{1}{n^3} S_{n,2} = \frac{1}{3}$$

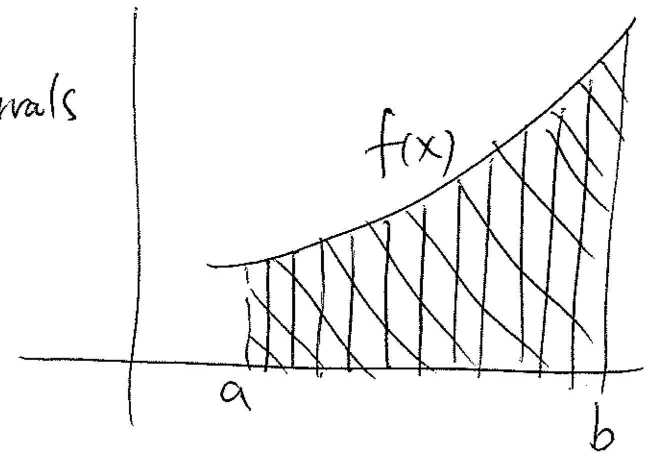
$$S_{n,3} \qquad \lim \frac{1}{n^4} S_{n,3}$$

Sum of geometric sequence

$$\sum_{k=1}^{n} q^k = \frac{q^{n+1}-q}{q-1}$$

The definite integral

Divide $[a, b]$ into n sub-intervals



$$\Delta x = \frac{(b-a)}{n}$$

$$S_n = \underline{\Delta x} \sum_{k=1}^{n} f(a + k\Delta x)$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$$

= Signed area between $y = f(x)$, $y = 0$

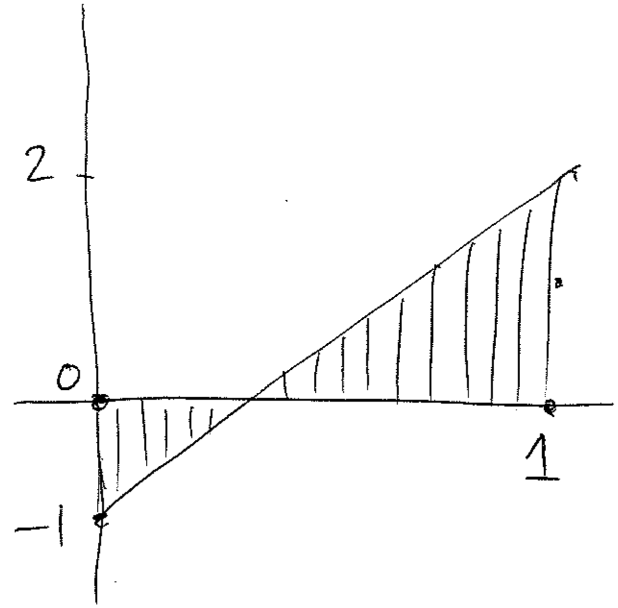
$x = a$ and $x = b$

Ex $\int_0^1 (3x-1)\,dx$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(3\cdot\frac{k}{n} - 1\right)$$

$$= \lim_{n\to\infty} \left(\frac{3}{n^2} \sum_{k=1}^{n} k - \frac{1}{n} \sum_{k=1}^{n} 1\right)$$

$$= 3 \times \frac{1}{2} - 1 = \frac{1}{2}$$



We need a more efficient method for evaluating definite integrals !

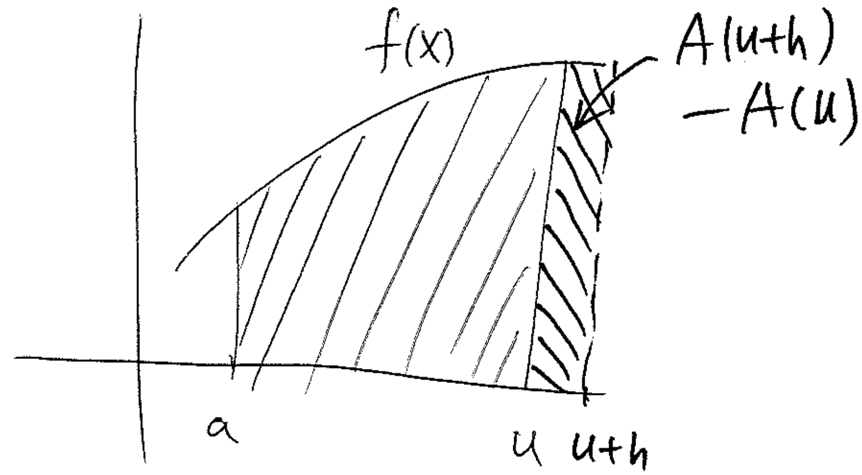
§14.7 The fundamental theorem of calculus

Goal: to evaluate $\int_a^b f(x)\,dx$

Consider $\quad A(u) = \int_a^u f(x)\,dx$

$= \text{Area between } y=f(x),\ y=0,\ x=a \text{ and } x=u$

We treat u as an independent variable.



$$A(u+h)-A(u) \approx h\cdot f(u)$$

$$\longrightarrow \quad \frac{A(u+h)-A(u)}{h} \approx f(u)$$

$$\longrightarrow \quad \lim_{h\to 0}\frac{A(u+h)-A(u)}{h} = f(u)$$

$$\longrightarrow \quad A'(u) = f(u)$$ with initial condition

$$A(a) = \int_a^a f(x)\,dx = 0$$

Let $F(x)$ be an anti-derivative of $f(x)$

(Think about $\int f(x)\,dx$)

$\rightarrow \quad (A(u) - F(u))' = 0$

$\rightarrow \quad \boxed{A(u) - F(u) = C}$

$A(a) = 0$

$\rightarrow \quad 0 - F(a) = C \quad \rightarrow \quad C = -F(a)$

$A(u) - F(u) = C$

$-F(a)$

$\rightarrow \quad \boxed{A(u) = F(u) - F(a)}$

Set $u = b$. $\quad \boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

Notation $\quad F(x)\Big|_a^b \equiv F(b) - F(a)$

Fundamental theorem of calculus

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$$

Ex $\int_1^3 (x^3+3x+2)\,dx$

$$\int (x^3+3x+2)\,dx = \frac{x^4}{4} + \frac{3}{2}x^2 + 2x + C$$

$$\int_1^3 (x^3+3x+2)\,dx = \left(\frac{x^4}{4} + \frac{3}{2}x^2 + 2x\right)\Big|_1^3$$

$$= \frac{1}{4}(3^4-1^4) + \frac{3}{2}(3^2-1^2) + 2(3-1) = \frac{1}{4}\times 80 + \frac{3}{2}\times 8 + 2\times 2 = \underline{36}$$

Ex $\int_1^4 \frac{1}{\sqrt{t}} e^{\sqrt{t}}\,dt$

$$= \left(2e^{\sqrt{t}}\right)\Big|_1^4$$

$$= 2\left(e^{\sqrt{4}} - e^{\sqrt{1}}\right)$$

$$= 2(e^2 - e)$$

$\int \frac{1}{\sqrt{t}} e^{\sqrt{t}}\,dt$ Change of variables

$u = \sqrt{t}$, $du = \frac{1}{2}\frac{1}{\sqrt{t}}\,dt$

$$= \int \underbrace{2e^{\sqrt{t}}}_{2e^u} \underbrace{\left(\frac{1}{2}\frac{1}{\sqrt{t}}\right)dt}_{du}$$

$$\frac{1}{\sqrt{t}} = 2\times\left(\frac{1}{2}\frac{1}{\sqrt{t}}\right)$$

$$= 2\int e^u\,du = 2e^u = 2e^{\sqrt{t}} + C$$

Ex: $\int_0^2 x\sqrt{3x^2+4}\,dx$

$= \frac{1}{9}\left(\sqrt{3x^2+4}\right)^3\Big|_0^2$

$= \frac{1}{9}\left[\left(\sqrt{3\times 2^2+4}\right)^3 - \left(\sqrt{3\times 0+4}\right)^3\right]$

$= \frac{1}{9}\left[(\sqrt{16})^3 - (\sqrt{4})^3\right]$

$= \frac{1}{9}(64-8) = \boxed{\frac{56}{9}}$

$\int x\sqrt{3x^2+4}\,dx$ Change of variables

$= \int \underbrace{\frac{1}{6}\sqrt{3x^2+4}}_{\frac{1}{6}\sqrt{u}}\,\underbrace{(6x)\,dx}_{du}$

$u = 3x^2+4$

$du = 6x\,dx$

$x = \frac{1}{6}(6x)$

$= \frac{1}{6}\int u^{\frac{1}{2}}\,du$

$= \frac{1}{6}\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{1}{9}u^{3/2} = \frac{1}{9}(3x^2+4)^{\frac{3}{2}} + C$

Properties of definite integral

*) $\int_a^a f(x)\,dx = 0$

*) For $b < a$, $\int_a^b f(x)\,dx \overset{\text{is defined as}}{=\!=} -\int_b^a f(x)\,dx$

Ex. $\int_2^1 f(x)\,dx = -\int_1^2 f(x)\,dx$

*) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

*). $\int_a^b (c_1 f_1(x) + c_2 f_2(x))\,dx = c_1 \int_a^b f_1(x)\,dx + c_2 \int_a^b f_2(x)\,dx$

*) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

"b" does not need to be between "a" and "c"

Ex. $\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$

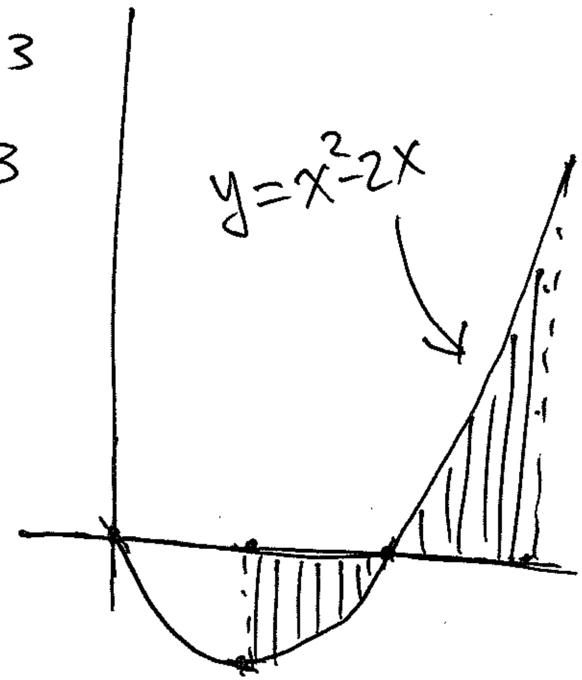
$\int_1^4 f(x)\,dx = \int_1^5 f(x)\,dx + \underbrace{\int_5^4 f(x)\,dx}_{-\int_4^5 f(x)\,dx}$

§14.9 Area between curves

Important: We find the absolute area, not the signed area

Ex. The area bounded by $y = x^2 - 2x$, $y = 0$

$x = 1$, and $x = 3$



$y(0) = 0$, $y(1) = -1$, $y(2) = 0$, $y(3) = 3$

$$y'(x) = 2x - 2 = 2(x-1) = \begin{cases} < 0, & x < 1 \\ > 0, & x > 1 \end{cases}$$

$$\text{Area} = \int_1^2 -(x^2 - 2x)\,dx + \int_2^3 (x^2 - 2x)\,dx$$

$$\int (x^2 - 2x)\,dx = \frac{x^3}{3} - x^2 + C$$

$$-\int_1^2 (x^2 - 2x)\,dx = -\left(\frac{x^3}{3} - x^2\right)\Big|_1^2 = -\left[\frac{1}{3}(2^3 - 1^3) - (2^2 - 1^2)\right]$$

$$= -\left[\frac{7}{3} - 3\right] = \frac{2}{3}$$

$$\int_2^3 (x^2-2x)\,dx = \left(\frac{x^3}{3} - x^2\right)\Big|_2^3 = \frac{1}{3}(3^3-2^3) - (3^2-2^2)$$

$$= \frac{1}{3}(19) - 5 = \frac{4}{3}$$

$$\text{Area} = \frac{2}{3} + \frac{4}{3} = 2$$

Ex: The area bounded by $y=3-x^2$ and $y=x^2+1$

Find the intersections of the two curves

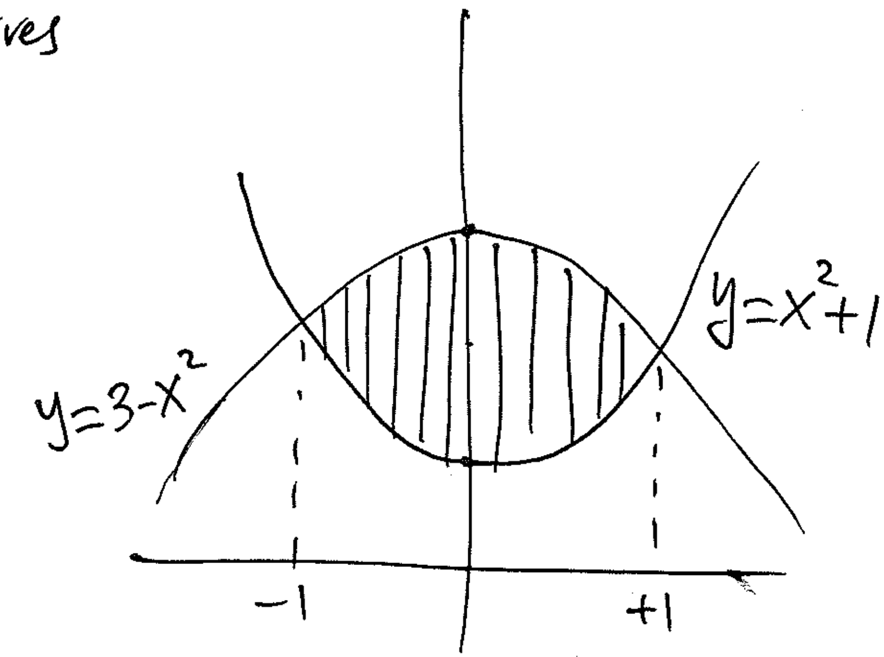
At an intersection,

$$3-x^2 = x^2+1$$

$$\rightarrow \quad 2 = 2x^2$$

$$\rightarrow \quad 1 = x^2$$

$$\rightarrow \quad x = \pm 1$$



$$\text{Area} = \int_{-1}^{1} \left((3-x^2) - (x^2+1) \right) dx = \int_{-1}^{1} (2-2x^2)\, dx$$

$$\int (2-2x^2)\, dx = 2x - \frac{2}{3}x^3 + C$$

$$\text{Area} = \left(2x - \frac{2}{3}x^3\right)\Big|_{-1}^{1} = 2\left(1-(-1)\right) - \frac{2}{3}\left(1^3 - (-1)^3\right)$$

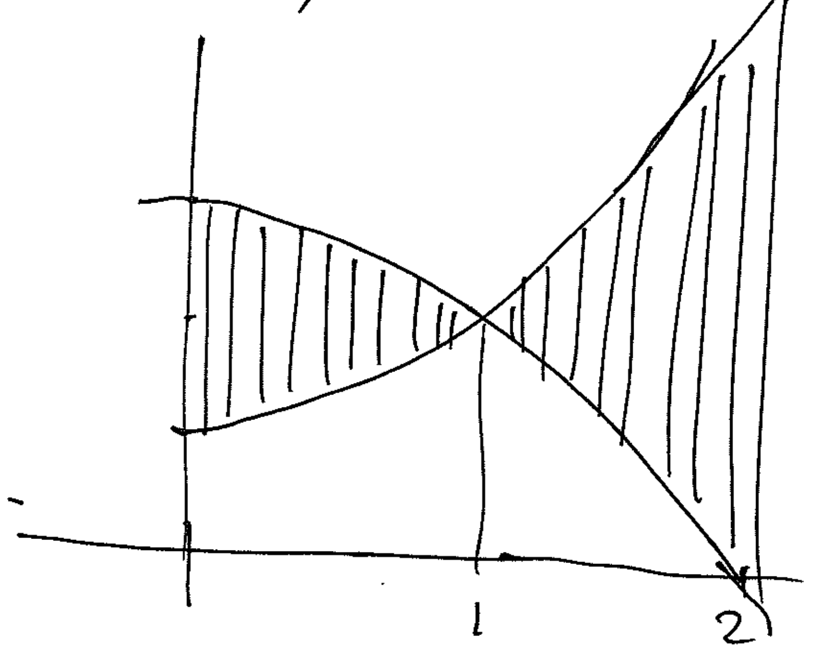
$$= 2 \times 2 - \frac{2}{3} \times 2 = 4 - \frac{4}{3} = \boxed{\frac{8}{3}}$$

<u>Ex</u>: The area bounded by $y = 3 - x^2$, $y = x^2 + 1$, $x = 0$ and $x = 2$

$$\text{Area} = \int_{0}^{1} \left((3-x^2) - (x^2+1) \right) dx$$

$$+ \int_{1}^{2} \left((x^2+1) - (3-x^2) \right) dx$$

$$= \int_{0}^{1} (2-2x^2)\, dx + \int_{1}^{2} -(2-2x^2)\, dx$$



$$\int_0^1 (2-2x^2)\,dx = \left(2x - \frac{2}{3}x^3\right)\Big|_0^1 = 2\times(1-0) - \frac{2}{3}(1-0)$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$

$$-\int_1^2 (2-2x^2)\,dx = -\left(2x - \frac{2}{3}x^3\right)\Big|_1^2 = -\left[2(2-1) - \frac{2}{3}(2^3 - 1^3)\right]$$

$$= -\left[2 - \frac{2}{3}\times 7\right] = -\left[\frac{6-14}{3}\right] = \frac{8}{3}$$

Area $= \frac{4}{3} + \frac{8}{3} = 4$