Recap: Change of variables

Given the task $\int g(x)\,dx$

Try $u = u(x)$, $du = u'(x)\,dx$

We write $\int g(x)\,dx = \int \underbrace{\frac{1}{u'(x)} g(x)}\, \underbrace{u'(x)\,dx}_{du}$

If this is a function of u, call it $f(u)$

$1 = \frac{1}{u'(x)} u'(x)$

$= \int f(u)\,du \Big|_{u=u(x)}$

$$\boxed{\int e^u\,du = e^u + C}$$

Ex. $\int (3x+1) e^{3x^2+2x}\,dx$

Try $u = 3x^2+2x$

$du = (6x+2)\,dx$

$3x+1 = \frac{1}{2}(6x+2)$

$= \int \underbrace{\frac{1}{2} e^{3x^2+2x}}_{\frac{1}{2}e^u} \underbrace{(6x+2)\,dx}_{du}$

$= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u = \boxed{\frac{1}{2}e^{3x^2+2x} + C}$

Summation notation

upper limit of summation

$$\sum_{k=m}^{n} A_k = \underbrace{A_m}_{k=m} + \underbrace{A_{m+1}}_{k=m+1} + \cdots + \underbrace{A_n}_{k=n}, \quad m \leq n$$

lower limit of summation

Summand.

Ex.

$$\sum_{k=-1}^{17} (2k+1) = \underbrace{-1}_{k=-1} + \underbrace{1}_{k=0} + \underbrace{3}_{k=1} + \cdots + \underbrace{35}_{k=17}$$

$$\sum_{i=3}^{10} (i^2+i-1) = \underbrace{(3^2+3-1)}_{i=3} + \underbrace{(4^2+4-1)}_{i=4} + \underbrace{(5^2+5-1)}_{i=5} + \cdots + \underbrace{(10^2+10-1)}_{i=10}$$

$$= 11 + 19 + 29 + \cdots + 109$$

Properties

*) $\displaystyle\sum_{K=m}^{n} (A_K + B_K) = \left(\sum_{K=m}^{n} A_K\right) + \left(\sum_{K=m}^{n} B_K\right)$

*) $\displaystyle\sum_{K=m}^{n} c\,A_K = c\left(\sum_{K=m}^{n} A_K\right)$

*) $\displaystyle\sum_{K=m}^{n} A_K = \sum_{K=m}^{\ell} A_K + \sum_{K=\ell+1}^{n} A_K \,, \qquad m \le \ell \le n .$

Summation formulas

*) $\displaystyle\sum_{K=1}^{n} 1 = \underbrace{\underbrace{1}_{K=1} + \underbrace{1}_{K=2} + \cdots + \underbrace{1}_{K=n}}_{n} = n .$

$\displaystyle \boxed{S_{n,0} = \sum_{K=1}^{n} 1 = n} \qquad \boxed{\sum_{K=1}^{n} c = c \cdot n}$

*) $\displaystyle S_{n,1} = \sum_{K=1}^{n} K = \underbrace{1}_{K=1} + \underbrace{2}_{K=2} + \cdots + \underbrace{n}_{K=n} \boxed{= \frac{n(n+1)}{2}}$

Derivation #1

$$S_{n,1} = 1 + 2 + \cdots + n$$
$$S_{n,1} = n + (n-1) + \cdots + 1$$

$$2\,S_{n,1} = \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n} = n(n+1)$$

$$\longrightarrow \boxed{S_{n,1} = \frac{n(n+1)}{2}}$$

Derivation #2

Consider $(k+1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1$

$$\longrightarrow \sum_{k=1}^{n} \left[(k+1)^2 - k^2 \right] = \sum_{k=1}^{n} (2k+1)$$

$k=1$: $\cancel{2}^2 - 1^2$

$k=2$: $\cancel{3}^2 - \cancel{2}^2$

$k=3$: $\cancel{4}^2 - \cancel{3}^2$

$$2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$k=n$: $(n+1)^2 - \cancel{n}^2$

$$\longrightarrow (n+1)^2 - 1 = 2\,S_{n,1} + S_{n,0}$$

n

$$n^2+2n = 2S_{n,1}+n.$$

$$\longrightarrow \quad n^2+n = 2S_{n,1}$$

$$\longrightarrow \quad \boxed{S_{n,1} = \frac{n^2+n}{2}}$$

$$\boxed{S_{n,0} = \sum_{k=1}^{n} 1 = n.}$$

$$S_{n,2} = \sum_{k=1}^{n} k^2 = 1^2+2^2+\cdots+n^2 = \boxed{\frac{2n^3+3n^2+n}{6}}$$

$$S_{n,3} = \sum_{k=1}^{n} k^3 = 1^3+2^3+\cdots+n^3 = \boxed{\frac{n^4+2n^3+n^2}{4}}$$

Ex. $\displaystyle\sum_{k=1}^{10}(k^2+2k+3) = \underbrace{\sum_{k=1}^{10} k^2}_{S_{n,2}} + 2\underbrace{\sum_{k=1}^{10} k}_{S_{n,1}} + 3\underbrace{\sum_{k=1}^{10} 1}_{S_{n,0}}$

$$= \frac{2\times 10^3+3\times 10^2+10}{6} + 2\times\frac{10^2+10}{2} + 3\times 10$$

$$= 385+110+30 = 525$$

Ex. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2} = \lim_{n\to\infty}\frac{1}{n^2}\underbrace{\sum_{k=1}^{n}k}_{S_{n,1}} = \lim_{n\to\infty}\frac{1}{n^2}\frac{(n^2+n)}{2}$$

$$= \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

$$\boxed{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2} = \frac{1}{2}}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3} = \lim_{n\to\infty}\frac{1}{n^3}\underbrace{\sum_{k=1}^{n}k^2}_{S_{n,2}} = \lim_{n\to\infty}\frac{1}{n^3}\frac{(2n^3+3n^2+n)}{6}$$

$$= \lim_{n\to\infty}\frac{1}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = \frac{2}{6} = \frac{1}{3}.$$

$$\boxed{\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2}{n^3} = \frac{1}{3}}$$

Sum of geometric sequence.

$$\boxed{\sum_{k=1}^{n} q^k = \frac{q^{n+1}-q}{q-1}}$$

Derivation:

$$(q-1)\,q^k = q^{k+1} - q^k.$$

$$\sum_{k=1}^{n}(q-1)\,q^k = \sum_{k=1}^{n}\left(q^{k+1}-q^k\right)$$

$$\underbrace{(q-1)\sum_{k=1}^{n} q^k}_{} = \begin{array}{ll} k=1: & \cancel{q^2} - q \\ k=2: & \cancel{q^3} - \cancel{q^2} \\ k=3: & \cancel{q^4} - \cancel{q^3} \\ & \\ k=n: & q^{n+1} - \cancel{q^n} \end{array}$$

$$(q-1)\sum_{k=1}^{n} q^k = q^{n+1} - q$$

$$\boxed{\sum_{k=1}^{n} q^k = \frac{q^{n+1}-q}{q-1}}$$

Ex. $\sum_{m=1}^{10}\left(\frac{2}{3}\right)^m$

$$\sum_{m=1}^{n}\left(\frac{2}{3}\right)^m=\frac{\left(\frac{2}{3}\right)^{n+1}-\left(\frac{2}{3}\right)}{\frac{2}{3}-1}=\frac{\frac{2}{3}-\left(\frac{2}{3}\right)^{n+1}}{\frac{1}{3}}=2-3\left(\frac{2}{3}\right)^{n+1}$$

$$\sum_{m=1}^{10}\left(\frac{2}{3}\right)^m=2-3\left(\frac{2}{3}\right)^{11}\approx 1.9653$$

Ex. $\sum_{j=9}^{25}3^j$

$$\sum_{j=1}^{n}3^j=\frac{3^{n+1}-3}{3-1}=\frac{1}{2}\left(3^{n+1}-3\right)$$

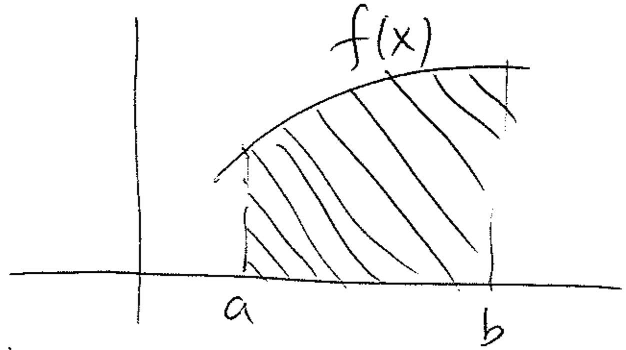
$$\sum_{j=1}^{25}=\sum_{j=1}^{8}+\sum_{j=9}^{25}$$

$$\Longrightarrow \sum_{j=9}^{25}3^j=\sum_{j=1}^{25}3^j-\sum_{j=1}^{8}3^j=\frac{1}{2}\left(3^{25+1}-3\right)-\frac{1}{2}\left(3^{8+1}-3\right)$$

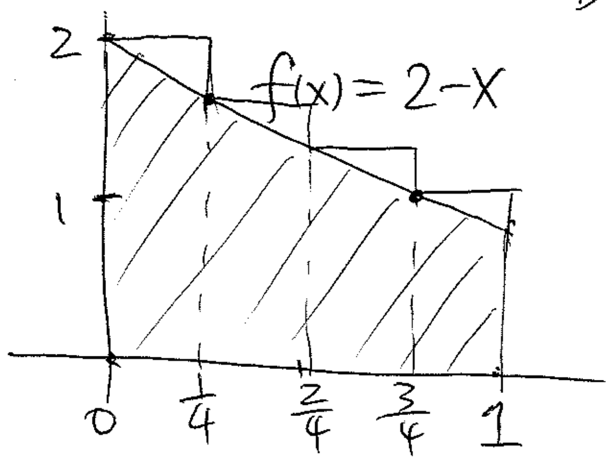
$$=\frac{1}{2}\left(3^{26}-3^9\right)$$

§14.6 The definite integral

Consider the area between the x-axis and f(x)



Ex



Divide $[0,1]$ into 4 subintervals

$\overline{S_4}$ = upper sum approximation of the area

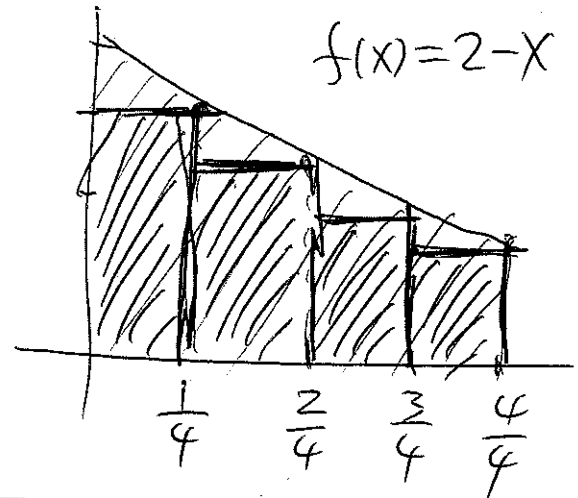
$$= \frac{1}{4} f\left(\frac{0}{4}\right) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{2}{4}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)$$

$$= \sum_{k=1}^{4} \frac{1}{4} f\left(\frac{k-1}{4}\right)$$

$\underline{S_4}$: $\underline{\text{lower sum}}$ approximation of the area

$$= \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{2}{4}\right) + \frac{1}{4}f\left(\frac{3}{4}\right) + \frac{1}{4}f\left(\frac{4}{4}\right)$$

$$= \sum_{k=1}^{4} \frac{1}{4} f\left(\frac{k}{4}\right)$$



Divide $[0,1]$ into n subintervals

$$\overline{S}_n = \cancel{\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k-1}{n}\right)} \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k-1}{n}\right)$$

$$\underline{S}_n = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right)$$

Use $\underline{f(x) = 2-x}$ and take the limit as $n \to \infty$

$$\overline{S}_n = \sum_{k=1}^{n} \frac{1}{n}\left(2 - \frac{k-1}{n}\right) = \frac{2}{n}\underbrace{\sum_{k=1}^{n} 1}_{S_{n,0}} - \frac{1}{n^2}\underbrace{\sum_{k=1}^{n} k}_{S_{n,1}} + \frac{1}{n^2}\underbrace{\sum_{k=1}^{n} 1}_{S_{n,0}}$$

Recall $\lim_{n\to\infty} \frac{1}{n} S_{n,0} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} 1 = 1$

$$\lim_{n\to\infty} \frac{1}{n^2} S_{n,1} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{1}{n^3} S_{n,2} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{3}$$

$$\lim_{n\to\infty} \overline{S}_n = 2 - \frac{1}{2} + 0 = \frac{3}{2}$$

$$\underline{S}_n = \sum_{k=1}^{n} \frac{1}{n}\left(2 - \frac{k}{n}\right) = \frac{2}{n} \underbrace{\sum_{k=1}^{n} 1}_{S_{n,0}} - \frac{1}{n^2} \underbrace{\sum_{k=1}^{n} k}_{S_{n,1}}$$

$$\lim_{n\to\infty} \underline{S}_n = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\boxed{\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \underline{S}_n}$$

When $\lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} \underline{S}_n$, the common limit is called the definite integral of $f(x)$ over $[a, b]$.

and is denoted by

upper limit of integration

$$\int_a^b f(x)\, dx$$

lower limit of integration

integrand

How to evaluate a definite integral?

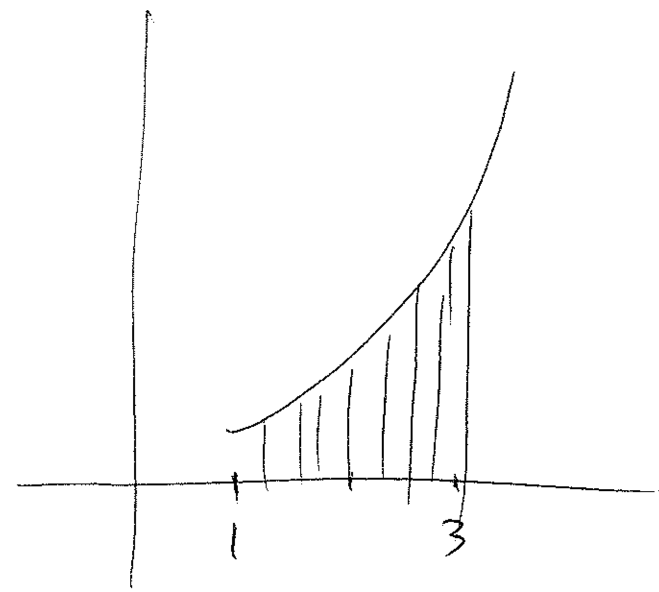
S_n = sum using the right-hand endpoint of each sub-interval

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} S_n .$$

Ex. $\int_1^3 x^2\,dx$, Size of each subinterval $= \frac{2}{n}$

The k-th subinterval

$$\left[1+(k-1)\frac{2}{n},\ \underline{1+k\frac{2}{n}}\right]$$



$$S_n = \sum_{k=1}^{n} \frac{2}{n} f\left(1+k\frac{2}{n}\right)$$

$$= \sum_{k=1}^{n} \frac{2}{n}\left(1+\frac{2k}{n}\right)^2$$

$$= \sum_{k=1}^{n} \frac{2}{n}\left(1+\frac{4k}{n}+\frac{4k^2}{n^2}\right)$$

$$= \frac{2}{n}\underbrace{\sum_{k=1}^{n} 1}_{S_{n,0}} + \frac{8}{n^2}\underbrace{\sum_{k=1}^{n} k}_{S_{n,1}} + \frac{8}{n^3}\underbrace{\sum_{k=1}^{n} k^2}_{S_{n,2}}$$

$$\lim_{n\to\infty} S_n = 2 + 8\times\frac{1}{2} + 8\times\frac{1}{3} = \frac{26}{3}$$