Recap Given $y'(x)$ and $y(0)$. Find $y(x)$

Revenue $r(q)$, $r(0)=0$

Marginal revenue (MR) $\frac{dr}{dq}$

Given $\frac{dr}{dq}$. Find $r(q)$ and $p(q)=\frac{r(q)}{q}$.

Demand function.

and $PED = \frac{dq}{dp} \cdot \frac{p}{q}$

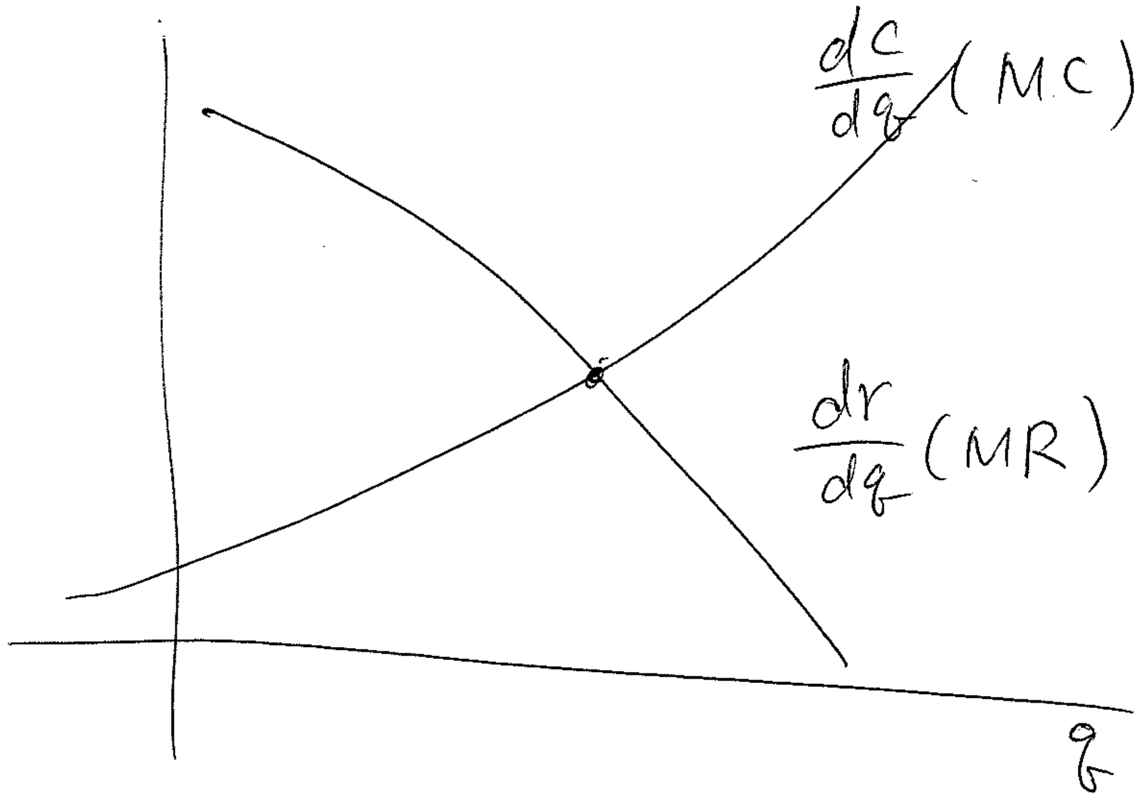
Total cost $C(q)$, fixed cost $C(0)$

Marginal cost (MC) $\frac{dC}{dq}$

Given $\frac{dC}{dq}$ and $C(0)$. Find $C(q)$

and $ATC = \frac{C(q)}{q}$

average total cost.


$\frac{dc}{dq}$ (MC)
$\frac{dr}{dq}$ (MR)
q

§14.4 Change of variables

Chain rule: $\underline{\left(F(u(x))\right)' = F'(u)\Big|_{u=u(x)} \cdot u'(x)}$

Let $f(u) = F'(u)$.

$$F(u) = \int f(u)\,du$$

Integrate: $\boxed{\int f(u)\Big|_{u=u(x)} u'(x)\,dx = F(u)\Big|_{u=u(x)} + C = \int f(u)\,du\Big|_{u=u(x)}}$

Given the task $\int g(x)\,dx$

If we can write $g(x) = f(u)\Big|_{u=u(x)} \cdot u'(x)$

then $\int g(x)\,dx = \underbrace{\int f(u)\,du}_{\text{Integral in new variable } u}\Big|_{u=u(x)}$

Procedure: Given the task $\int g(x)\,dx$

*) Identify and try $u(x)$

*) write out $du = u'(x)\,dx$

*) Examine whether or not we can write

$$g(x)\,dx = f(u)\,du$$

*) If yes, then

$$\int g(x)\,dx = \int f(u)\,du \Big|_{u=u(x)}$$

Ex. $\int 6(2x+3)^{-5}dx$. Try $u = 2x+3$, $du = 2\,dx$. $\int u^{-5}du$.

$$= \int \underbrace{\frac{6}{2}(2x+3)^{-5}}_{3u^{-5}} \underbrace{2\,dx}_{du}$$

$$= 3\int u^{-5}du = 3\,\frac{u^{-5+1}}{-5+1} = -\frac{3}{4}u^{-4}$$

$$\boxed{= -\frac{3}{4}(2x+3)^{-4} + C}$$

Recall $\int x^{\alpha}dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C$, $\alpha \neq -1$

Ex $\int (2x^2+1)(2x^3+3x+5)^3\,dx$ Try $u = 2x^3+3x+5$

$du = (6x^2+3)\,dx$

$$= \int \underbrace{\frac{1}{3}(2x^3+3x+5)^3}_{\frac{1}{3}u^3} \underbrace{(6x^2+3)\,dx}_{du}$$

$$= \frac{1}{3}\int u^3\,du = \frac{1}{3}\cdot\frac{u^{3+1}}{3+1} = \frac{1}{12}u^4 = \frac{1}{12}(2x^3+3x+5)^4 + C$$

$$(2x^2+1) = \frac{6x^2+3}{3}$$

It does not always work!

<u>Ex</u>. $\int x(x^3+1)\,dx$ Try $u=x^3+1$

$du=3x^2dx$

$$=\int \underbrace{\frac{1}{3x}(x^3+1)}\ \underbrace{3x^2dx}_{du}$$

$$x=\frac{3x^2}{3x}$$

<u>It does not work.</u>

Find a different way.

$$\int x(x^3+1)\,dx=\int(x^4+x)\,dx=\underline{\frac{x^5}{5}+\frac{x^2}{2}+C}$$

<u>Ex</u>. $\int \frac{5x}{1-3x^2}dx$ Try $u=1-3x^2$

$du=(-6x)dx$

$$=\int \underbrace{\frac{5}{(-6)}\frac{1}{(1-3x^2)}}_{\left(-\frac{5}{6}\right)\frac{1}{u}}\ \underbrace{(-6x)\,dx}_{du}$$

$$5x=\frac{5}{(-6)}(-6x)$$

Recall

$$\int\frac{1}{u}du=\ln|u|+C$$

$$=-\frac{5}{6}\int\frac{1}{u}du=-\frac{5}{6}\ln|u|=\underline{-\frac{5}{6}\ln|1-3x^2|+C}$$

7

Ex. $\int x^3 e^{1-x^4}\,dx$. Try $u=1-x^4$

$du=(-4x^3)dx$

$=\int \underbrace{\frac{-1}{4}e^{1-x^4}}_{-\frac{1}{4}e^u}\underbrace{(-4x^3)dx}_{du}$

$x^3=\left(\frac{-1}{4}\right)(-4x^3)$

$=-\frac{1}{4}\int e^u\,du=-\frac{1}{4}e^u=\boxed{-\frac{1}{4}e^{1-x^4}+C}$

$\boxed{\text{Recall } \int e^{au}\,du=\frac{1}{a}e^{au}+C}$

§14.5 More techniques.

$\int \frac{1}{ax+b}dx \quad (a\neq 0)$

Try $u=ax+b$

$du=a\,dx$

$1=\frac{1}{a}\times a$

$=\int \underbrace{\frac{1}{a}\frac{1}{(ax+b)}}_{\frac{1}{a}\cdot\frac{1}{u}}\cdot\underbrace{a\,dx}_{du}$

$=\frac{1}{a}\int\frac{1}{u}du=\frac{1}{a}\ln|u|=\frac{1}{a}\ln|ax+b|+C$

$\boxed{\int\frac{1}{ax+b}dx=\frac{1}{a}\ln|ax+b|+C}$

*) Reduce to a proper rational function.

$$\boxed{\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C}$$

Ex. $\displaystyle\int \frac{2x^3 - x + 2}{(x-2)}\,dx$

Long division.

$$
\begin{array}{r|l}
 & 2x^2 + 4x + 7 \\
\hline
x-2 & 2x^3 + 0\cdot x^2 - x + 2 \\
 & 2x^3 - 4x^2 \\
\hline
 & \quad 4x^2 - x + 2 \\
 & \quad 4x^2 - 8x \\
\hline
 & \qquad 7x + 2 \\
 & \qquad 7x - 14 \\
\hline
 & \qquad\quad 16.
\end{array}
$$

$$\int \frac{2x^3 - x + 2}{x-2}\,dx = \int \left(2x^2 + 4x + 7 + \frac{16}{x-2}\right) dx$$

$$= \frac{2}{3}x^3 + 2x^2 + 7x + 16\ln|x-2| + C$$

$\dfrac{P(x)}{Q(x)}$ is a proper rational function if degree $P(x) <$ degree $Q(x)$

Ex $\int b^x dx$. We write $b = e^{\ln b}$

$b^x = e^{(\ln b)x}$

$= \int e^{(\ln b)x} dx$

$= \frac{1}{\ln b} e^{(\ln b)x} + C$

$\boxed{\text{Recall } \int e^{ax} dx = \frac{1}{a} e^{ax} + C}$

$= \frac{1}{\ln b} b^x + C$

$$\boxed{\int b^x dx = \frac{1}{\ln b} b^x + C}$$

Ex $\int 3^{2-5x} dx$

Try $u = 2-5x$

$du = (-5)dx$

$1 = \left(\frac{-1}{5}\right) \times (-5)$

$= \int \underbrace{\left(\frac{-1}{5}\right) 3^{2-5x}}_{\left(-\frac{1}{5}\right)3^u} \underbrace{(-5)dx}_{du}$

$= -\frac{1}{5}\int 3^u du = -\frac{1}{5}\frac{1}{\ln 3} 3^u = \boxed{-\frac{1}{5}\frac{1}{\ln 3} 3^{2-5x} + C}$

Ex. $\displaystyle\int \frac{x}{(x^2+1)\ln(x^2+1)}\,dx$ Try $u=\ln(x^2+1)$

$$= \int \underbrace{\frac{1}{2}\,\frac{1}{\ln(x^2+1)}}_{\frac{1}{2}\frac{1}{u}.}\cdot\underbrace{\frac{1}{x^2+1}\,2x\,dx}_{du}$$

$du = \dfrac{1}{x^2+1}\,2x\,dx$

$x = \frac{1}{2}\times(2x)$

$$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \boxed{\frac{1}{2}\ln\left|\ln(x^2+1)\right| + C}$$

Alternatively Try $u=(x^2+1)$

$$= \frac{1}{2}\int \frac{1}{u}\,\frac{1}{\ln u}\,du. \qquad \text{Try } v=\ln u.$$

$$= \frac{1}{2}\int \frac{1}{v}\,dv$$

Ex. $\cancel{\int \frac{x^2 e^{x^3}}{(e^{x^3}+1)^{1/2}} dx}$

Try $u = e^{x^3} + 1$

$du = e^{x^3}(3x^2)\,dx$

$x^2 = \frac{1}{3} \times (3x^2)$

$$\int \frac{x^2 e^{x^3}}{(e^{x^3}+1)^{1/3}}\,dx$$

$$= \int \underbrace{\frac{1}{3}\frac{1}{(e^{x^3}+1)^{1/3}}}_{\frac{1}{3}u^{-\frac{1}{3}}} \underbrace{e^{x^3}(3x^2)\,dx}_{du}$$

Recall $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$
$(\alpha \neq -1)$

$$= \frac{1}{3}\int u^{-\frac{1}{3}}\,du = \frac{1}{3}\cdot\frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{1}{3}\frac{u^{\frac{2}{3}}}{\frac{2}{3}} = \frac{1}{2}u^{\frac{2}{3}}$$

$$\boxed{= \frac{1}{2}(e^{x^3}+1)^{2/3} + C}$$

Consumption function, marginal propensity to consume (MPC)

$C(I)$

Consumption Income.

$$MPC : \frac{dC}{dI}$$

Given. $\frac{dC}{dI} = \frac{4}{5} + \frac{1}{10\sqrt{I+20}}$ and $C=4$ at $I=5$

Find $C(I)$.

$$C(I) = \int \left(\frac{4}{5} + \frac{1}{10\sqrt{I+20}}\right) dI$$

Try $u = I+20$.
$du = dI$.

$$= \int \left(\frac{4}{5} + \frac{1}{10} u^{-\frac{1}{2}}\right) du$$

$$= \frac{4}{5}u + \frac{1}{10} \cdot \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{4}{5}u + \frac{1}{5}u^{\frac{1}{2}} = \frac{4}{5}(I+20) + \frac{1}{5}(I+20)^{\frac{1}{2}}$$

$$= \frac{4}{5}I + \frac{1}{5}\sqrt{I+20} + \underline{\underline{K}}$$

$C = 4$ at $I = 5$

$\rightarrow \quad 4 = \frac{4}{5} \times 5 + \frac{1}{5}\sqrt{5+20} + K$

$\rightarrow \quad 4 = 4 + 1 + K$

$\rightarrow \quad K = -1$

$\rightarrow \quad \boxed{C(I) = \frac{4}{5} I + \frac{1}{5}\sqrt{I+20} - 1}$

Integration formulas.

$$\int k\,dx = kx + C$$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

$$\int b^{x}\,dx = \frac{1}{\ln b} b^{x} + C$$

$b = e^{\ln b}$

$b^{x} = e^{(\ln b)x}$