Review (continued)

Exponential decay $y(t) = y(0)\, e^{-\lambda t}$

Logistic growth $\frac{dy}{dt} = ry\left(1 - \frac{y}{M}\right)$

M: Carrying capacity

$\rightarrow \quad y(t) = \frac{M}{1+be^{-rt}} = \frac{M}{1+ba^t}, \quad a = e^{-r}$

Given M, $y(t)$ has 2 parameters a and b.

We need 2 data points to determine a and b.

Ex. Suppose $y(t)$ follows $\frac{dy}{dt} = ry\left(1 - \frac{y}{M}\right)$

with $M = 612$

$y(0) = 36$

$y(1) = 68$

Task: Find $y(4)$

$$y(0) = 36 \longrightarrow \frac{612}{1+b} = 36 \longrightarrow b = \frac{612}{36} - 1 = 16$$

$$y(1) = 68 \longrightarrow \frac{612}{1+16a} = 68 \longrightarrow 16a = \frac{612}{68} - 1 = 8$$

$$\longrightarrow a = \frac{1}{2}$$

$$y(t) = \frac{612}{1 + 16\left(\frac{1}{2}\right)^t}$$

$$y(4) = \frac{612}{1 + 16\left(\frac{1}{2}\right)^4} = \frac{612}{1+1} = 306$$

Newton's law of cooling

$$\frac{dT}{dt} = k\,(T(t) - a)$$

a: ambient temperature

$$\longrightarrow \quad \ln(T(t) - a) = kt + C$$

<u>Given a</u>, $T(t)$ has 2 parameters k and C

we need 2 data points to determine k and C.

Ex Suppose $T(t)$ follows $\frac{dT}{dt} = k(T(t) - a)$

with $a = 21$

$T(0) = 25$

$T(1) = 23$

Task: Find t_d such that $T(t_d) = 37$

At $t=0$, $\ln(25-21) = k \cdot 0 + C$

$\longrightarrow \quad C = \ln 4 = 2\ln 2$

At $t=1$, $\ln(23-21) = k \cdot 1 + C$

$\longrightarrow \quad \ln 2 = k + 2\ln 2 \quad \longrightarrow \quad k = -\ln 2$

$\ln(T(t) - 21) = -(\ln 2)\, t + 2\ln 2$

At t_d $\quad \underbrace{\ln(37-21)}_{\ln 16 = 4\ln 2} = -(\ln 2)\, t_d + 2\ln 2$

$\longrightarrow \quad 2\ln 2 = -(\ln 2)\, t_d$

$\longrightarrow \quad t_d = -2$

Partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

Ex. $g(x,y) = \sqrt{x^2 y^3 + 1}$

Chain rule $\quad u = x^2 y^3 + 1$

$$g(u) = \sqrt{u}$$

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{1}{2} u^{-\frac{1}{2}} \cdot 2 x y^3$$

$$= \frac{x y^3}{\sqrt{x^2 y^3 + 1}}$$

$$\frac{\partial g}{\partial y} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} = \frac{1}{2} u^{-\frac{1}{2}} \, 3 x^2 y^2$$

$$= \frac{3}{2} \frac{x^2 y^2}{\sqrt{x^2 y^3 + 1}}$$

Taylor polynomials for $f(x,y)$

$$T_1(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$T_2(x,y) = T_1(x,y) + \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2$$

Maxima and minima of $f(x,y)$

Critical point (a,b)

$$\begin{cases} f_x(a,b) = 0 \\ f_y(a,b) = 0 \end{cases}$$

Rule 1: If (a,b) is a maximum (or a minimum) then (a,b) is a critical point.

Rule 2 (Second derivative test)

Let (a,b) be a critical point

let $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

*) $D(a,b) > 0$ and $f_{xx} > 0$

$\longrightarrow$ (a,b) is a minimum

Think about $f(x,y) = (x-a)^2 + (y-b)^2$

*) $D(a,b) > 0$ and $f_{xx} < 0$

$\longrightarrow$ (a,b) is a maximum

Think about $f(x,y) = -(x-a)^2 - (y-b)^2$

*) $D(a,b) < 0$ $\longrightarrow$ (a,b) is a saddle point.

Think about $f(x,y) = (x-a)^2 - (y-b)^2$

*) $D(a,b) = 0$ $\longrightarrow$ It is inconclusive.

Ex. $f(x,y) = x^3 - \frac{3}{2}y^2 + 3xy$

Step 1. Find critical points

$$f_x = 3x^2 + 3y = 0 \quad (1)$$

$$f_y = -3y + 3x = 0 \quad (2)$$

$(2) \rightarrow -y + x = 0 \rightarrow y = x$

$(1) \rightarrow x^2 + y = 0 \rightarrow x^2 + x = 0 \rightarrow x(x+1) = 0$

$$\rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = -1 \\ y = -1 \end{cases}$$

Two critical points: $(0,0)$, $(-1,-1)$

Step 2. Second derivative test

$$f_{xx} = 6x, \quad f_{xy} = 3, \quad f_{yy} = -3$$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 6x(-3) - 3^2 = -18x - 9$$

At $(0,0)$

$D(0,0) = -9 < 0 \longrightarrow (0,0)$ is a saddle point.

At $(-1,-1)$

$D(-1,-1) = -18(-1) - 9 = 18 - 9 = 9 > 0$

$f_{xx} = 6(-1) = -6 < 0$

$\longrightarrow (-1,-1)$ is a maximum.

Dependence of maximum value on parameters

$F(x, y; \alpha, \beta)$ $\quad (x, y)$: variables

$\quad (\alpha, \beta)$: parameters

Let $(x^*(\alpha,\beta), y^*(\alpha,\beta))$ be the maximum point

and $F^*(\alpha,\beta)$ be the maximum value of $F(x, y; \alpha, \beta)$

$$F^*(\alpha,\beta) = F(x^*(\alpha,\beta), y^*(\alpha,\beta); \alpha, \beta)$$

Suppose we are give x^*, y^* and F^* at (α_0, β_0)

Task: Estimate $F^*(\alpha_0+\Delta\alpha, \beta_0+\Delta\beta)$

Method: Linear approximation

$$\Delta F^* = F^*(\alpha_0+\Delta\alpha, \beta_0+\Delta\beta) - F^*(\alpha_0, \beta_0)$$

$$\approx \frac{\partial F^*}{\partial \alpha} \cdot \Delta\alpha + \frac{\partial F^*}{\partial \beta} \Delta\beta$$

Envelope theorem

$$\frac{\partial F^*}{\partial \alpha} = \left.\frac{\partial F}{\partial \alpha}\right|_{(x,y)=(x^*,y^*)}$$

$$\frac{\partial F^*}{\partial \beta} = \left.\frac{\partial F}{\partial \beta}\right|_{(x,y)=(x^*,y^*)}$$

Ex. $F(x,y;\alpha,\beta) = 14x + 10y - \left(3 + \frac{\alpha-3}{1+y^2}\right)x^2 - (\beta-2)y^2$
$- 2xy + 20$

We are given that at $(\alpha_0, \beta_0) = (3, 5)$

$x^*(3,5) = 2$

$y^*(3,5) = 1$

$F^*(3,5) = 39$

Task: Estimate $F^*(3.1, 5.3)$

$\Delta\alpha = 0.1, \quad \Delta\beta = 0.3$

$$\frac{\partial F^*}{\partial \alpha} = \left.\frac{\partial F}{\partial \alpha}\right|_{(x^*,y^*)} = -\left.\frac{x^2}{1+y^2}\right|_{(2,1)} = \frac{-4}{1+1} = -2$$

$$\frac{\partial F^*}{\partial \beta} = \left.\frac{\partial F}{\partial \beta}\right|_{(x^*,y^*)} = -\left.y^2\right|_{(2,1)} = -1$$

$$\Delta F^* = F^*(3.1,\ 5.3) - F^*(3,\ 5)$$

$$\approx \frac{\partial F^*}{\partial \alpha}\Delta\alpha + \frac{\partial F^*}{\partial \beta}\Delta\beta = (-2)(0.1) + (-1)(0.3) = -0.5$$

$$\longrightarrow \quad F^*(3.1,\ 5.3) \approx 39 + (-0.5) = 38.5$$

Lagrange multiplier

Ex. Maximize $U(x,y) = 20x + 14y - 2x^2 - y^2$

Subject to $3x + 2y = I$

Solve the problem for $I = 12$

$$F(x,y,\lambda) = \left(20x + 14y - 2x^2 - y^2\right) - \lambda(3x + 2y - 12)$$

$$F_x = 20 - 4x - 3\lambda = 0 \qquad (1)$$

$$F_y = 14 - 2y - 2\lambda = 0 \qquad (2)$$

$$F_\lambda = -(3x + 2y - 12) = 0 \qquad (3)$$

$$(1) \longrightarrow 4x = 20 - 3\lambda \longrightarrow x = 5 - \frac{3}{4}\lambda$$

$$(2) \longrightarrow 2y = 14 - 2\lambda \longrightarrow y = 7 - \lambda$$

$$(3) \longrightarrow 3x + 2y = 12 \longrightarrow 3 \times \left(5 - \frac{3}{4}\lambda\right) + 2(7 - \lambda) = 12$$

$$\longrightarrow 15 - \frac{9}{4}\lambda + 14 - 2\lambda = 12$$

$$\longrightarrow 15 + 14 - 12 = \frac{9+8}{4}\lambda$$

$$\longrightarrow 17 = \frac{17}{4}\lambda \longrightarrow \boxed{\lambda^* = 4}$$

$$x^* = 5 - \frac{3}{4}\lambda^* = 5 - \frac{3}{4} \times 4 = 2$$

$$y^* = 7 - \lambda^* = 7 - 4 = 3$$

$$U^* = U(x^*, y^*) = 20 \times 2 + 14 \times 3 - 2 \times 2^2 - 3^2$$

$$= 40 + 42 - 8 - 9 = 65$$

$$\boxed{\begin{array}{l} \lambda^* = 4 \\ x^* = 2 \\ y^* = 3 \\ U^* = 65 \end{array}}$$

Meaning of λ^*

$$\frac{dU^*}{dI} = \lambda^*$$

$$\Delta U^* = U^*(I_0 + \Delta I) - U^*(I_0) \approx \lambda^* \Delta I$$

Ex. Now suppose budget I increases from 12 to 14.

Task: Estimate $U^*(14)$

$$\Delta I = 14 - 12 = 2$$

$$\Delta U^* = U^*(14) - U^*(12) \approx \lambda^* \Delta I = 4 \times 2 = 8$$

$$U^*(14) \approx U^*(12) + 8 = 65 + 8 = 73$$

About the final exam

*) No computer, no phone, no calculator allowed.

*) closed book, closed note

*) One-sheet note is allowed, both sides

*) Duration of final exam: 2 hours

*) # of problems: 8 ~ 10