Recap

Maximize $U(x,y,z)$ subject to budget $xP_x + yP_y + zP_z = I$

$$F(x,y,z,\lambda) = U(x,y,z) - \lambda(xP_x + yP_y + zP_z - I)$$

$$F_x = U_x - \lambda P_x = 0 \quad (1) \longrightarrow \lambda = \frac{U_x}{P_x}$$

U_x ← MU per unit of x

$\frac{U_x}{P_x}$ — MU per \$ of x

$$F_y = U_y - \lambda P_y = 0 \quad (2) \longrightarrow \lambda = \frac{U_y}{P_y}$$

$\frac{U_y}{P_y}$ — MU per \$ of y

$$F_z = U_z - \lambda P_z = 0 \quad (3) \longrightarrow \lambda = \frac{U_z}{P_z}$$

$\frac{U_z}{P_z}$ — MU per \$ of z

$$F_\lambda = -(xP_x + yP_y + zP_z - I) = 0 \quad (4)$$

At the maximum utility

$$\lambda^* = \frac{U_x}{P_x} = \frac{U_y}{P_y} = \frac{U_z}{P_z}$$

λ^* — MU per \$ of x, y, z, or any combination

Meaning of λ^*

let $U^*(I)$ be the maximum ~~utility~~ $U(x,y,z)$

subject to $xP_x + yP_y + zP_z = I$

We have $\dfrac{dU^*(I)}{dI} = \lambda^*$

$$\Delta U^* = U^*(I + \Delta I) - U^*(I) \approx \lambda^* \Delta I$$

Ex. Maximize $U(x,y,z) = 5\ln x + 8\ln y + 12 \ln z$

subject to $10x + 15y + 30z = I$

We are given that at $I_0 = 3000$

$$(x^*, y^*, z^*, \lambda^*) = \left(60, 64, 48, \frac{1}{120}\right)$$

$$U^*(I_0) = U(x^*, y^*, z^*) = 100.2$$

Task: Estimate $U^*(3600)$

$$I_0 = 3000, \quad I_0 + \Delta I = 3600 \quad \rightarrow \Delta I = 600$$

$$\Delta U^* = U^*(3600) - U^*(3000) \approx \lambda^* \Delta I = \frac{1}{120} \times 600 = 5$$

$$U^*(3600) \approx U^*(3000) + 5 = \underline{105.2}$$

$\boxed{\text{Review}}$ Differential: $df = f'(x)\,dx$

Approximation: $\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\,\Delta x$

Indefinite integral $\int f(x)\,dx = F(x) + C$

where $F'(x) = f(x)$

$$\int e^{ax}\,dx = \frac{1}{a} e^{ax} + C$$

$$\int x^{\alpha}\,dx = \frac{1}{\alpha+1} x^{\alpha+1} + C \qquad (\alpha \neq -1)$$

Ex. $\int (t^4 + e^{3t})\,dt = \int t^4\,dt + \int e^{3t}\,dt$

$$= \frac{1}{5} t^5 + \frac{1}{3} e^{3t} + C$$

Chain rule: $f(u(x))$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Change of variables

$$\int g(x)\,dx \qquad \text{Try } u(x), \quad du = u'(x)\,dx$$

$$= \int \underbrace{g(x)\frac{1}{u'(x)}}\,\underbrace{u'(x)\,dx}_{du} = \int f(u)\,du$$

If this is a nice function of u ...

Ex. $\int (x^2+2x)\, e^{x^3+3x^2}\,dx$

Try $u = x^3+3x^2$, $du = (3x^2+6x)\,dx = 3(x^2+2x)\,dx$

$$= \int \underbrace{\frac{1}{3} e^{x^3+3x^2}}_{\frac{1}{3}e^u}\,\underbrace{3(x^2+2x)\,dx}_{du}$$

$$= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3+3x^2} + C$$

$$\int (ax+b)^\alpha dx = \frac{1}{a}\frac{1}{\alpha+1}(ax+b)^{\alpha+1} + C, \quad a \neq 0, \quad \alpha \neq -1$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + C, \quad a \neq 0$$

Ex $$\int \frac{1}{\sqrt{3x+4}} dx = \int (3x+4)^{-\frac{1}{2}} dx = \frac{1}{3}\cdot\frac{1}{-\frac{1}{2}+1}(3x+4)^{-\frac{1}{2}+1} + C$$

$$= \frac{2}{3}\sqrt{3x+4} + C$$

$$\int b^x dx = \int e^{\ln(b)x} dx = \frac{1}{\ln(b)} e^{\ln(b)x} + C$$

$$= \frac{1}{\ln(b)} b^x + C$$

Long division: $$\int \frac{x^3+x^2+2}{x+2} dx$$

$$= \int \left(x^2 - x + 2 - \frac{2}{x+2}\right) dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 2x - 2\ln|x+2| + C$$

$$\begin{array}{r} x^2 - x + 2 \\ (x+2)\overline{\big)\, x^3 + x^2 + 0\cdot x + 2} \\ \underline{x^3 + 2x^2 } \\ -x^2 + 0\cdot x + 2 \\ \underline{-x^2 - 2x } \\ 2x + 2 \\ \underline{2x + 4} \\ -2 \end{array}$$

Integration with initial condition.

Ex. $f'(x) = \dfrac{1}{\sqrt{3x+4}}$, $f(0)=2$. Find $f(x)$

$$f(x) = \int f'(x)\,dx$$

$$= \frac{2}{3}\sqrt{3x+4} + C$$

$$2 = \frac{2}{3}\sqrt{3\times 0+4} + C$$

$$\rightarrow 2 = \frac{2}{3}\sqrt{4} + C$$

$$\rightarrow 2 = \frac{4}{3} + C \rightarrow C = \frac{2}{3}$$

$$\rightarrow \boxed{f(x) = \frac{2}{3}\sqrt{3x+4} + \frac{2}{3}}$$

Summation

$$S_{n,0} = \sum_{k=1}^{n} 1 = n$$

$$S_{n,1} = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} q^k = \frac{q^{n+1} - q}{q-1}$$

The definite integral

The fundamental theorem of calculus

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$$

Ex. $\displaystyle\int_0^2 \frac{x}{\sqrt{3x^2+4}}\,dx$

Try $u = 3x^2+4$, $du = 6x\,dx$

$$\int \frac{x}{\sqrt{3x^2+4}}\,dx = \int \underbrace{\frac{1}{6}\frac{1}{\sqrt{3x^2+4}}}_{\frac{1}{6}u^{-\frac{1}{2}}}\underbrace{6x\,dx}_{du} = \frac{1}{6}\int u^{-\frac{1}{2}}\,du$$

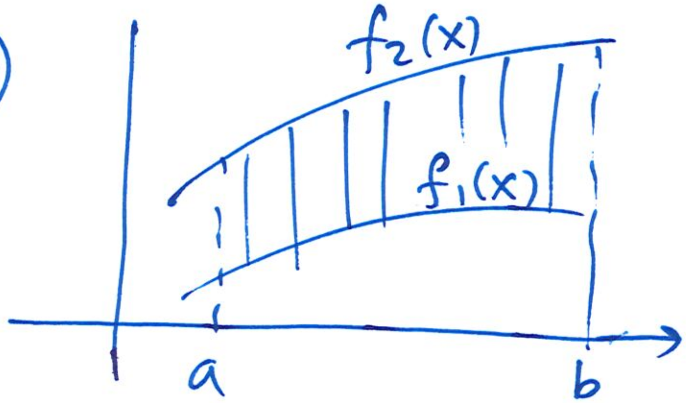
$$= \frac{1}{6}\cdot\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1} = \frac{1}{3}u^{\frac{1}{2}} = \frac{1}{3}\sqrt{3x^2+4}+C$$

$$\int_0^2 \frac{x}{\sqrt{3x^2+4}}\,dx = \frac{1}{3}\sqrt{3x^2+4}\Big|_0^2 = \frac{1}{3}\left(\sqrt{3\times 2^2+4}-\sqrt{4}\right)$$

$$= \frac{1}{3}\left(\sqrt{16}-\sqrt{4}\right) = \frac{2}{3}$$

Area between two curves

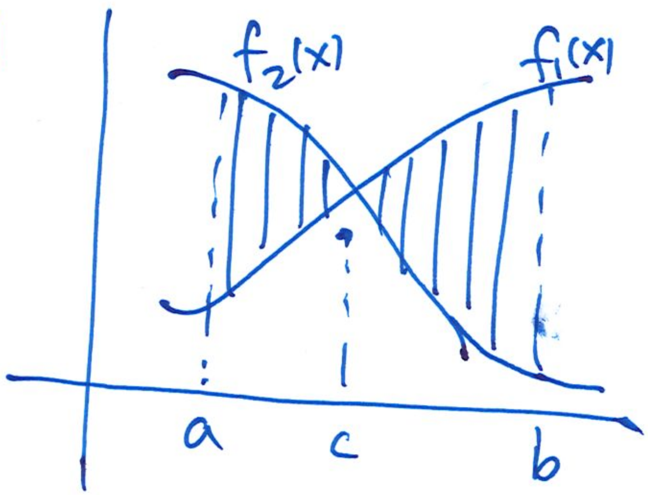
*)



a and b are given

$$\text{Area} = \int_a^b \left(f_2(x) - f_1(x)\right) dx$$

*)

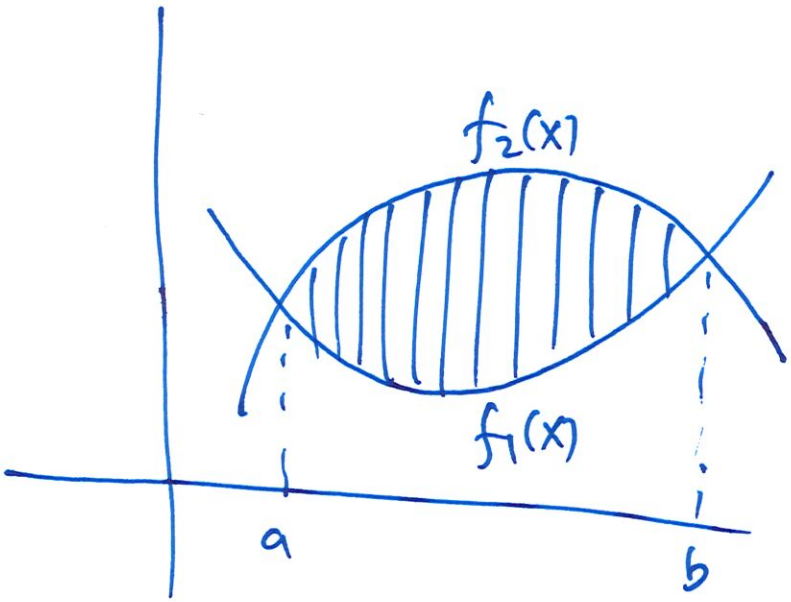


a and b are given

Find c

$$\text{Area} = \int_a^c \left(f_2(x) - f_1(x)\right) dx + \int_c^b \left(f_1(x) - f_2(x)\right) dx$$

*)

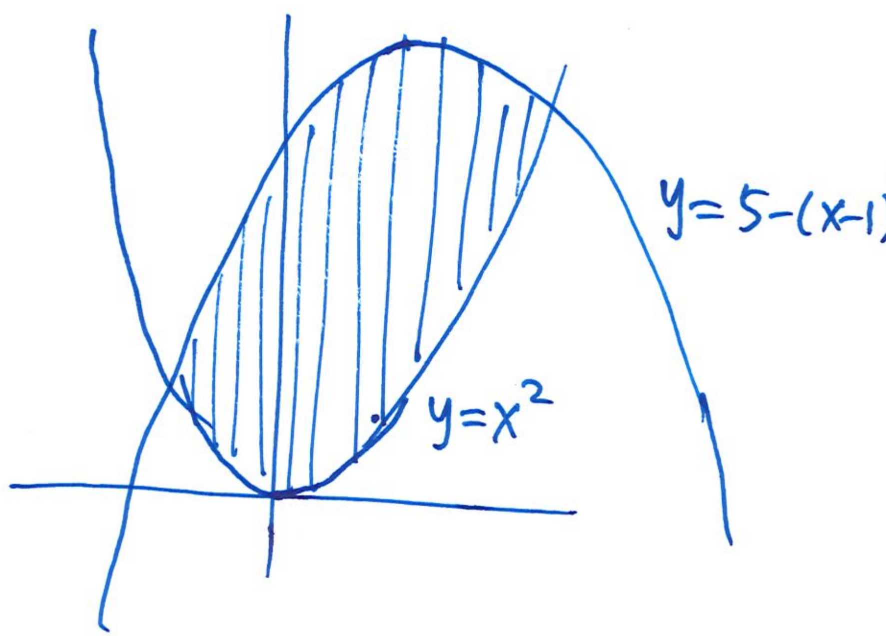


Find a and b

$$\text{Area} = \int_a^b (f_2(x) - f_1(x))\,dx$$

Ex. The area bounded by $y = x^2$ and $y = 5-(x-1)^2$

First, we find the intersections of the two curves.



$$x^2 = 5-(x-1)^2$$

$$\rightarrow x^2 + (x-1)^2 - 5 = 0$$

$$\rightarrow x^2 + (x^2 - 2x + 1) - 5 = 0$$

$\cancel{\rightarrow 2x^2+2x}$

$\rightarrow 2x^2-2x-4=0$

$\rightarrow x^2-x-2=0$

$\rightarrow (x-2)(x+1)=0$

$\rightarrow x=-1 \quad \text{or} \quad x=2$

Two intersections: $(-1, 1)$, $(2, 4)$

$$\text{Area} = \int_{-1}^{2} \left(5-(x-1)^2-x^2\right) dx$$

$$= \left(5x - \frac{1}{3}(x-1)^3 - \frac{1}{3}x^3\right)\Big|_{-1}^{2}$$

$$= 5\times[2-(-1)] - \frac{1}{3}[1^3-(-2)^3] - \frac{1}{3}[2^3-(-1)^3]$$

$$= 5\times 3 - \frac{1}{3}\times 9 - \frac{1}{3}\times 9 = 15-3-3 = 9$$

Product rule: $(u(x)\,v(x))' = u(x)\,v'(x) + u'(x)\,v(x)$

Integration by parts

$$\int u(x)\underbrace{v'(x)\,dx}_{dv} = u(x)v(x) - \int v(x)\underbrace{u'(x)\,dx}_{du}.$$

A concise form:

$$\int u\,dv = u\cdot v - \int v\,du$$

Ex. $\int (t+2)e^{3t}\,dt$

Try $u = t+2$, $v'(t) = e^{3t}$

$\to v(t) = \int e^{3t}\,dt = \frac{1}{3}e^{3t}$

$$= (t+2)\cdot\frac{1}{3}e^{3t} - \int \frac{1}{3}e^{3t}\,dt$$

$$= \frac{t+2}{3}e^{3t} - \frac{1}{3}\cdot\frac{1}{3}e^{3t} + C = \left(\frac{t+2}{3} - \frac{1}{9}\right)e^{3t} + C$$

Partial fraction:

Ex. $\displaystyle\int \frac{x^3+x}{x^2+2x-3}\,dx$

Long division:

$$\longrightarrow \frac{x^3+x}{x^2+2x-3} = (x-2) + \underbrace{\frac{8x-6}{x^2+2x-3}}_{(x+3)(x-1)}$$

$$\frac{8x-6}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

$$= \frac{A(x-1)+B(x+3)}{(x+3)(x-1)}$$

$$\rightarrow 8x-6 = A(x-1) + B(x+3)$$

At $x=-3 \longrightarrow -30 = A(-4) \rightarrow A = \frac{15}{2}$

At $x=1 \longrightarrow 2 = B\cdot(4) \longrightarrow B = \frac{1}{2}$

$$\int \frac{x^3+x}{x^2+2x-3}\,dx = \int\left((x-2)+\frac{15}{2}\frac{1}{x+3}+\frac{1}{2}\frac{1}{x-1}\right)dx$$

$$= \frac{1}{2}(x-2)^2+\frac{15}{2}\ln|x+3|+\frac{1}{2}\ln|x-1|+C$$

Integration by tables

Ex. $\displaystyle\int \frac{1}{\sqrt{1+2t^2}}\,dt = \int \frac{1}{\sqrt{2}\sqrt{t^2+\frac{1}{2}}}\,dt$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}}\,dt$$

Item 27: $\displaystyle\int \frac{1}{\sqrt{u^2\pm a^2}}\,du = \ln\left|u+\sqrt{u^2\pm a^2}\right|+C$

$$= \frac{1}{\sqrt{2}}\ln\left|t+\sqrt{t^2+\frac{1}{2}}\right|+C$$

Differential eq, separation of variables

$$\frac{dy}{dx} = -3(y-1)x^2$$

$$\int \frac{dy}{y-1} = -\int 3x^2 dx$$

$$\ln|y-1| = -x^3 + C_1$$

$$|y-1| = e^{C_1} e^{-x^3}$$

$$(y-1) = \pm e^{C_1} e^{-x^3} = C e^{-x^3}$$

$$y = C e^{-x^3} + 1$$

Additional condition: $\boxed{y(0) = \frac{1}{2}}$

$$\frac{1}{2} = C e^0 + 1 \longrightarrow \frac{1}{2} = C + 1 \longrightarrow C = -\frac{1}{2}$$

$$y(x) = 1 - \frac{1}{2} e^{-x^3}$$

Exponential decay

$$y(t) = y(0)\, e^{-\lambda t}$$

$\lambda =$ decay rate

half life: t_h

$$e^{-\lambda t_h} = \frac{1}{2}$$

$$-\lambda t_h = -\ln(2)$$

$$t_h = \frac{\ln(2)}{\lambda}$$

$$\lambda = \frac{\ln(2)}{t_h}.$$

Ex Decay of ^{14}C

$y(t) = y(0)\, e^{-\lambda t}$ has a half life $t_h = 5730$ years

Find decay rate λ.

$$\lambda = \frac{\ln(2)}{t_h} = \frac{\ln(2)}{5730} \Big/ \text{year}$$

Ex At an unknown t_1, $y(t_1) = 75\%$ of $y(0)$

Task: Estimate t_1

At t_1.

$$\underbrace{y(t_1)}_{0.75\, y(0)} = y(0)\, e^{-\lambda t_1}$$

$$\rightarrow \quad 0.75 = e^{-\lambda t_1}$$

$$\rightarrow \quad \ln(0.75) = -\lambda t_1$$

$$\rightarrow \quad t_1 = \frac{-\ln(0.75)}{\lambda} = -\frac{\ln(0.75)}{\ln(2)}\, 5730 \text{ years}.$$