Constrained optimization and the envelope theorem

1. A monopolistic firm sells one product in two markets, A and B. The daily demand equations for the firm's product in these markets are given by

$$Q_A = 100 - 0.4P_A \qquad \text{and} \qquad Q_B = 120 - 0.5P_B,$$

where Q_A and Q_B are the demands and P_A and P_B are the prices for the firm's product in markets A and B, respectively. The firm's constant marginal cost is \$40 and the its daily fixed cost is \$2500.

- **a.** Find the prices that the firm should charge in each market to maximize its daily profit. Use the second derivative test to verify that the prices you found yield the *absolute* maximum profit.
- **b.** Use the *envelope theorem* (and linear approximation) to estimate the change in the firm's max profit if the marginal cost of their product increases to \$40.75.

2. Jack's (gustatory) utility function is

$$U(x, y, z) = 5\ln x + 7\ln y + 18\ln z,$$

where x is the number of fast-food meals Jack consumes in a month; y is the number of 'diner' meals he consumes in a month; and z is the number of 'fancy restaurant' meals he consumes in a month.

The average price of a fast-food meal is $p_x = \$4.00$; the average price of a 'diner' meal is $p_y = \$8.00$; and the average price of a 'fancy restaurant' meal is $p_z = \$30.00$.

- **a.** How many meals of each type should Jack consumer per month to maximize his utility, if his monthly budget for these meals is $\beta = \$1200.00$?
- **b.** By approximately how much will Jack's utility increase if his budget increases by \$50.00? Explain your answer.

3. A firm's production function is given by

$$Q = 10K^{0.4}L^{0.7},$$

where Q is the firm's annual output, K is the annual capital input, and L is the annual labor input. The cost per unit of capital is \$1000, and the cost per unit of labor is \$4000.

- **a.** Find the levels of labor and capital inputs that **minimize** the cost of producing an output of $Q = 20,000$ units.
- **b.** Find the levels of labor and capital inputs that **minimize** the cost of producing an output of $Q = q$ units. Express your answer in terms of q.

4. Consider the function $F(x, y) = 3x^2 - Axy + By^2 - 2x - 4y + 5$, with *variables* x and y and *parameters* A and B.

- **(a)** Find the critical point and critical value of this function when $A = A_0 = 12$ and $B = B_0 = 19$.
- **(b)** Use the ***envelope theorem*** and *linear approximation* to predict how the critical value will change if the parameter A changes from $A_0 = 12$ to $A_1 = 12.5$ and the parameter B changes from $B_0 = 19$ to $B_1 = 19.2$.

5. The annual output for a luxury hotel chain is given by $Q = 30K^{2/5}L^{1/2}R^{1/4}$, where K, L and R are the capital, labor and real estate inputs, all measured in \$1,000,000 s, and Q is the average number of rooms rented per day.

The hotel chain's annual budget is $B = \$69$ million.

a. How should they allocate this budget to the three inputs in order to *maximize* their annual output? What is the maximum output?

b. What is the critical value of the multiplier when output is maximized?

c. Use your answer to **b.** to compute the *approximate* change in the firm's maximum output if their annual budget increases by $\$500,000$? Explain your answer.