Recap.

*) Differential

$$dy = f'(x)\,dx$$

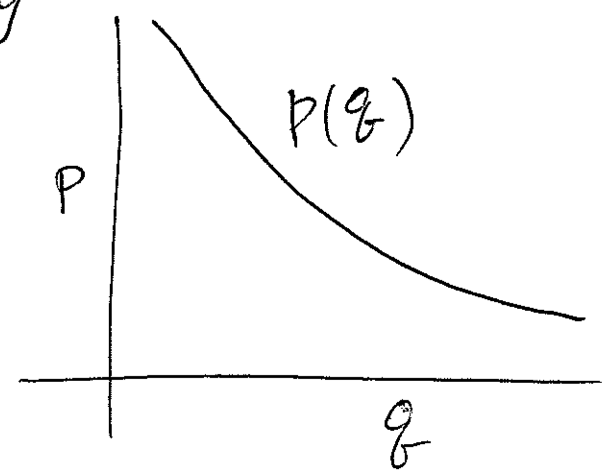
Infinitesimal change in y

Infinitesimal change in x

We use $dy = f'(x)dx$ with finite dx to approximate the true Δy

*) Demand function $P(q)$

$$PED = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{dq}{dp}\cdot\frac{p}{q}$$



*) Antiderivative of $f(x)$

$$F'(x) = f(x)$$

Indefinite integral of $f(x)$

$$\int f(x)\,dx = F(x) + C.$$

Integration formulas. $\int k\,dx = kx + C$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln x + C$$

$$\int e^{x}\,dx = e^{x} + C$$

§14.3 Integration with initial conditions.

Suppose we know $f'(x)$

$$f(x) = \int f'(x)\,dx$$

Ex $f'(x) = 2x^2$

$$f(x) = \int 2x^2\,dx = 2\,\frac{x^{2+1}}{2+1} = \frac{2}{3}x^3 + C$$

If $f(1) = 2$, then

$$f(x)\Big|_{x=1} = \left(\frac{2}{3}x^3 + C\right)\Big|_{x=1}$$

$$\Rightarrow \quad 2 = \frac{2}{3} + C \quad \Rightarrow \quad C = \frac{4}{3}$$

$$\Rightarrow \boxed{f(x) = \frac{2}{3}x^3 + \frac{4}{3}}$$

Ex. Given $y'(t) = \frac{1}{\sqrt[3]{t}} + e^{-2t}$ and $y(0) = 2$. Find $y(t)$

$$y(t) = \int \left(\frac{1}{\sqrt[3]{t}} + e^{-2t} \right) dt$$

$$\int t^{-\frac{1}{3}} dt = \frac{t^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} = \frac{3}{2} t^{\frac{2}{3}} + C$$

Recall $\frac{d}{dt} e^{at} = e^{at} \cdot a. \longrightarrow \frac{d}{dt}\left(\frac{1}{a} e^{at} \right) = e^{at}$

$$\longrightarrow \boxed{\int e^{at} dt = \frac{1}{a} e^{at} + C} \quad \int e^{ax} dx = \frac{1}{a} e^{ax} + C$$

$$\int e^{-2t} dt = \frac{1}{-2} e^{-2t} = -\frac{1}{2} e^{-2t} + C$$

$$\boxed{e^{0} = 1}$$

$$\longrightarrow y(t) = \frac{3}{2} t^{\frac{2}{3}} - \frac{1}{2} e^{-2t} + C$$

$$y(0) = 2 \longrightarrow 2 = -\frac{1}{2} + C \longrightarrow C = \frac{5}{2}$$

$$\boxed{y(t) = \frac{3}{2} t^{\frac{2}{3}} - \frac{1}{2} e^{-2t} + \frac{5}{2}}$$

Revenue and marginal revenue

$$\underbrace{r}_{\text{revenue}} = \underbrace{p}_{\text{price}} \times \underbrace{q}_{\text{quantity}}$$

initial condition

$$r(q) = P(q) \cdot q \qquad \boxed{r(0) = 0}$$

$$\underbrace{\frac{dr}{dq}}_{\text{marginal revenue}}$$

meaning = change in revenue corresponding to unit change in quantity demanded.

In general, $\frac{dr}{dq} \neq P$.

Given marginal revenue. Find <u>revenue</u> and <u>demand function</u>. (6)

<u>Ex</u> Given $\frac{dr}{dq} = 1000 - 10q - q^2$, Find $r(q)$ and $p(q)$

$$r(q) = \int (1000 - 10q - q^2)\, dq = 1000q - 10\frac{q^2}{2} - \frac{q^3}{3} + C$$

$$r(0) = 0 \longrightarrow C = 0$$

$$\boxed{r(q) = 1000q - 5q^2 - \frac{1}{3}q^3}$$

$$p(q) = \frac{r(q)}{q} = \frac{1000q - 5q^2 - \frac{1}{3}q^3}{q} = 1000 - 5q - \frac{1}{3}q^2$$

Total cost (TC), marginal cost (MC), fixed cost

$$\underbrace{c(q)}_{\text{Total cost}} = \underbrace{\text{fixed cost}}_{\text{independent of } q} + \underbrace{\text{variable cost}}_{\substack{=0 \\ \text{when } q=0}}$$

$C(0) = \text{fixed cost}$ ← initial condition

$\underbrace{\dfrac{dC}{dq}}_{\text{marginal cost}}$ meaning = change in cost corresponding to unit change in quantity produced.

Given marginal cost and fixed cost. Find the total cost (TC) and find the average ~~total~~ cost (ATC)

Ex. Given $\frac{dC}{dq} = 0.003q^2 - 0.4q + 40$

and $\boxed{C(0) = 5000}$

Find $\underbrace{C(q)}_{TC}$ and $\underbrace{\frac{C(q)}{q}}_{ATC}$

$$C(q) = \int (0.003q^2 - 0.4q + 40)\,dq$$

$$= 0.003\frac{q^3}{3} - 0.4\frac{q^2}{2} + 40q + C$$

$C(0) = 5000 \longrightarrow C = 5000$

$$\underbrace{C(q)}_{TC} = 0.001q^3 - 0.2q^2 + 40q + 5000$$

Average total cost (ATC) $= \dfrac{C(q)}{q}$

$$= 0.001q^2 - 0.2q + 40 + \frac{5000}{q}$$

At $q = 100$, $\dfrac{C(100)}{100} = 0.001 \times (100)^2 - 0.2 \times (100) + 40 + \dfrac{5000}{100}$

$$= 10 - 20 + 40 + 50 = \underline{\underline{80}}$$

(ATC)

We can integrate $y''(x)$ twice to find $y(x)$

Ex. Given $y''(x) = 2e^{-x} + 3$, $y'(0) = 7$, $y(0) = 5$. Find $y(x)$

$$y'(x) = \int y''(x)\,dx = \int (2e^{-x} + 3)\,dx$$

$$\boxed{\int e^{ax}dx = \frac{1}{a}e^{ax} + c}$$

$$= 2 \cdot \frac{e^{-x}}{-1} + 3x + c$$

$$\boxed{e^0 = 1}$$

$$y'(0) = 7 \longrightarrow 7 = -2 + c$$

$$\longrightarrow c = 9$$

$$\boxed{y'(x) = -2e^{-x} + 3x + 9}$$

$$y(x) = \int y'(x)\,dx = \int (-2e^{-x} + 3x + 9)\,dx$$

$$= -2 \cdot \frac{e^{-x}}{-1} + 3\frac{x^2}{2} + 9x + c$$

$$y(0) = 5 \longrightarrow 5 = 2 + c \longrightarrow c = 3$$

$$\boxed{y(x) = 2e^{-x} + \frac{3}{2}x^2 + 9x + 3}$$

$$\underbrace{\frac{dr}{dq}}_{\text{marginal revenue}} \rightarrow \underbrace{r(q)}_{\text{revenue}} \rightarrow \underbrace{p(q)}_{\text{demand function}} \rightarrow \text{PED}.$$

price elasticity of demand

Ex. Given. $\frac{dr}{dq} = 1000 - 10q - q^2$

$$\rightarrow r(q) = 1000q - 5q^2 - \frac{1}{3}q^3$$

$$\rightarrow p(q) = \frac{r(q)}{q} = 1000 - 5q - \frac{1}{3}q^2$$

$$\rightarrow \boxed{\text{PED} = \frac{dq}{dp} \cdot \frac{p}{q}}$$

$$\frac{dp}{dq} = -5 - \frac{2}{3}q$$

$$\frac{dq}{dp} = \frac{1}{\frac{dp}{dq}} = \frac{1}{-(5+\frac{2}{3}q)}$$

$$PED = \frac{dq}{dp} \cdot \frac{p}{q} = \frac{1}{-(5+\frac{2}{3}q)} \frac{(1000-5q-\frac{1}{3}q^2)}{q}$$

$$(30)^2 = 900$$

$$= \frac{-(1000-5q-\frac{1}{3}q^2)}{(5q+\frac{2}{3}q^2)}$$

At $q=30$.

$$PED = \frac{-(1000-150-300)}{(150+600)} = \frac{-550}{750}$$

$$= \frac{-55}{75} = \frac{-11}{15} \approx -0.73$$

$P \nearrow 1\%$, $q \searrow 0.73\%$

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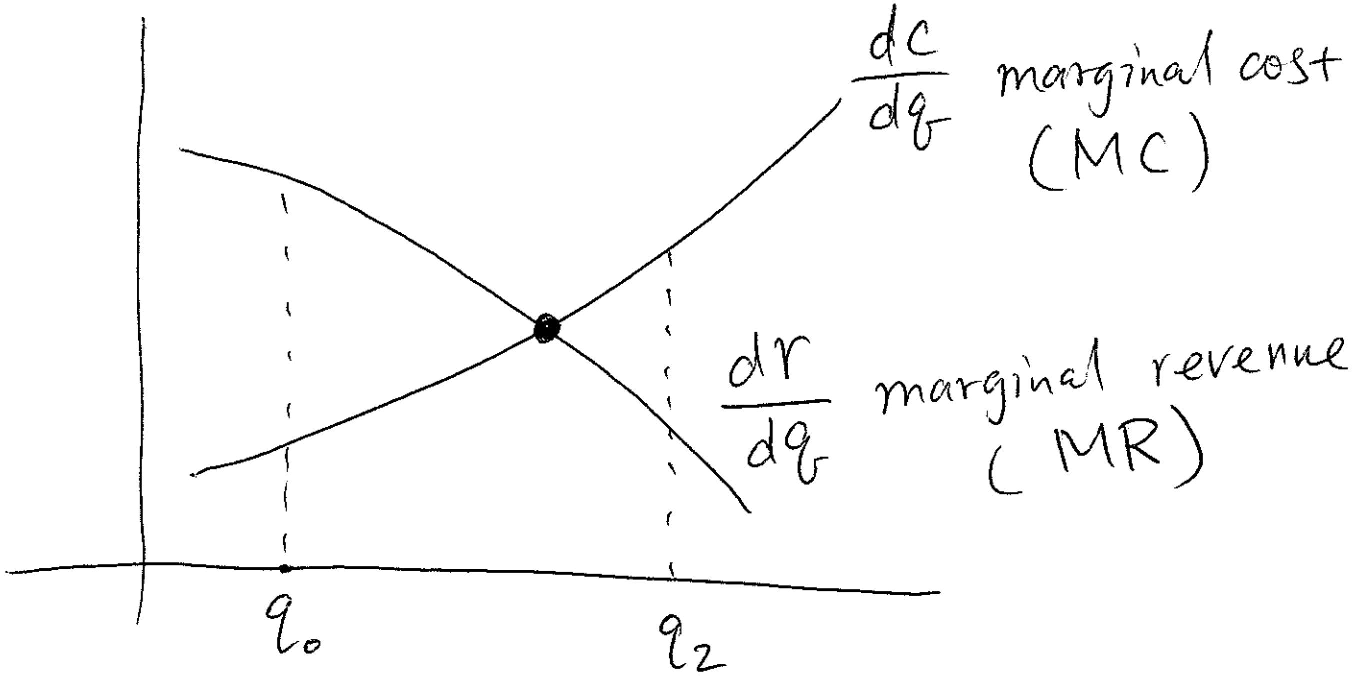
How does a company determine the quantity to produce? (13)



Review of Integration formulas

Chain rule: $f(u(x))' = f'(u)\,u'(x)$

$$\boxed{\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}.}$$

Ex. $e^{3x} = e^{u}$, $u = 3x$

$$\frac{d}{dx}(e^{3x}) = \frac{de^{u}}{du} \cdot \frac{du}{dx} = e^{u} \cdot 3 = 3e^{3x}$$

$$\boxed{\frac{d}{dx}(e^{ax}) = ae^{ax}}$$

$$\rightarrow \frac{d}{dx}\left(\frac{1}{a}e^{ax}\right) = e^{ax}$$

$$\rightarrow \boxed{\int e^{ax}dx = \frac{1}{a}e^{ax} + C}$$

$$\frac{dx}{dx} = 1 \longrightarrow \int 1\, dx = x + C$$

$$\int k\, dx = kx + C$$

$$\frac{d\, x^{\alpha+1}}{dx} = (\alpha+1)\, x^{\alpha} \longrightarrow \frac{d}{dx}\left(\frac{x^{\alpha+1}}{\alpha+1}\right) = x^{\alpha} \quad (\alpha \neq -1)$$

$$\longrightarrow \int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1)$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\longrightarrow \int \frac{1}{x} dx = \ln x + C$$

$$\int (f_1(x) + f_2(x))\, dx = \int f_1(x)\, dx + \int f_2(x)\, dx$$

$$\int k f_1(x)\, dx = k \int f_1(x)\, dx$$

$$\frac{d\, y(x)}{dx} \xleftarrow[\text{meaning}]{} y'(x)$$