§14.1 Differential

Difference

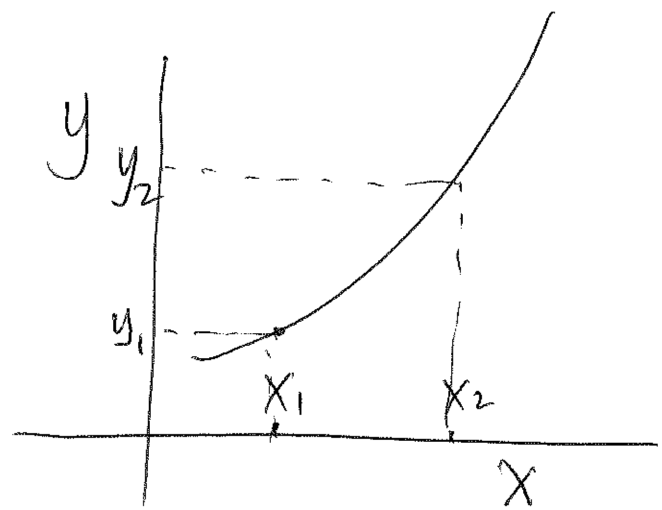
Consider two values of x

$x_1, \ x_2$

$\Delta x = x_2 - x_1$

A difference in x

Consider $y = f(x)$



$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$

The corresponding difference in y

$$\lim_{x_2 \to x_1} \frac{\Delta y}{\Delta x} = f'(x_1) \quad \text{derivative of } f(x)$$

$$\Rightarrow \quad \frac{\Delta y}{\Delta x} \approx f'(x_1) \quad \text{for small } \Delta x = x_2 - x_1$$

$$\Rightarrow \quad \Delta y \approx f'(x_1)\,\Delta x$$

Differential is defined as

Δy corresponding to infinitesimal Δx.

$$\boxed{dy = f'(x)\,dx}$$

We write the derivative as $\quad f'(x) = \dfrac{dy}{dx}$

In applications, we use

$$dy = f'(x)\,dx \quad \text{with finite } dx$$

to approximate the true Δy

Ex. $$y(x) = \frac{400}{\sqrt{x}+10}.$$

$$y'(x) = 400 \cdot \frac{-1}{(\sqrt{x}+10)^2} \cdot \frac{1}{2}\frac{1}{\sqrt{x}}$$

Recall: power rule. $(x^\alpha)' = \alpha x^{\alpha-1}$

chain rule. $f(u(x))' = f'(u)\,u'(x)$

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$f(u) = \frac{400}{u+10} \qquad u = \sqrt{x}$$

$$f'(u) = \frac{400(-1)}{(u+10)^2}, \quad u'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$$

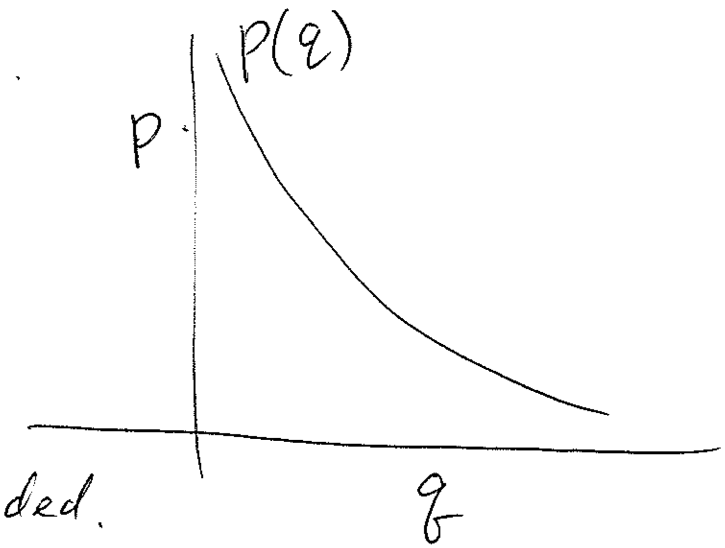
$$y'(x) = -200 \frac{1}{(\sqrt{x}+10)^2} \frac{1}{\sqrt{x}}$$

$$dy = y'(x)\,dx = -200 \frac{1}{(\sqrt{x}+10)^2} \frac{1}{\sqrt{x}}\,dx$$

Ex. Use ~~the~~ the differential to estimate a function value.

Consider the demand function.

$$P(q) = \frac{400}{\sqrt{q}+10}$$



q = quantity demanded.

P = the price to generate that quantity demanded.

At $q = 100$, $P(100) = 20$

We like to estimate $P(90)$

$$\left.\frac{dp}{dq}\right|_{q=100} = p'(q)\Big|_{q=100} = -200\frac{1}{(\sqrt{q}+10)^2}\frac{1}{\sqrt{q}} = -200\frac{1}{(20)^2}\frac{1}{10} = \frac{-1}{20}$$

$q=100$

$= -0.05$

$100 \to 90 \qquad \Delta q = -10$

$$dp = p'(q)\,dq = 0.5$$

$-0.05 \qquad -10$

approximation for the true Δp.

$$\boxed{\begin{aligned}\text{True } \Delta p &= p(90) - p(100)\\ &= 0.5267\end{aligned}}$$

$$p(90) = p(100) + \Delta p \simeq p(100) + dp = 20 + 0.5 = \underline{20.5}$$

<u>Ex</u>. $\displaystyle \frac{dq}{dp} = \frac{1}{\left(\frac{dp}{dq}\right)} = -\frac{(\sqrt{q}+10)^2\sqrt{q}}{200}$.

<u>At $q=100$, $p=20$</u> $\displaystyle \frac{dq}{dp} = -\frac{(\sqrt{100}+10)^2\sqrt{100}}{200} = -20$

We like to estimate $q(21)$

$q(20) = 100$.

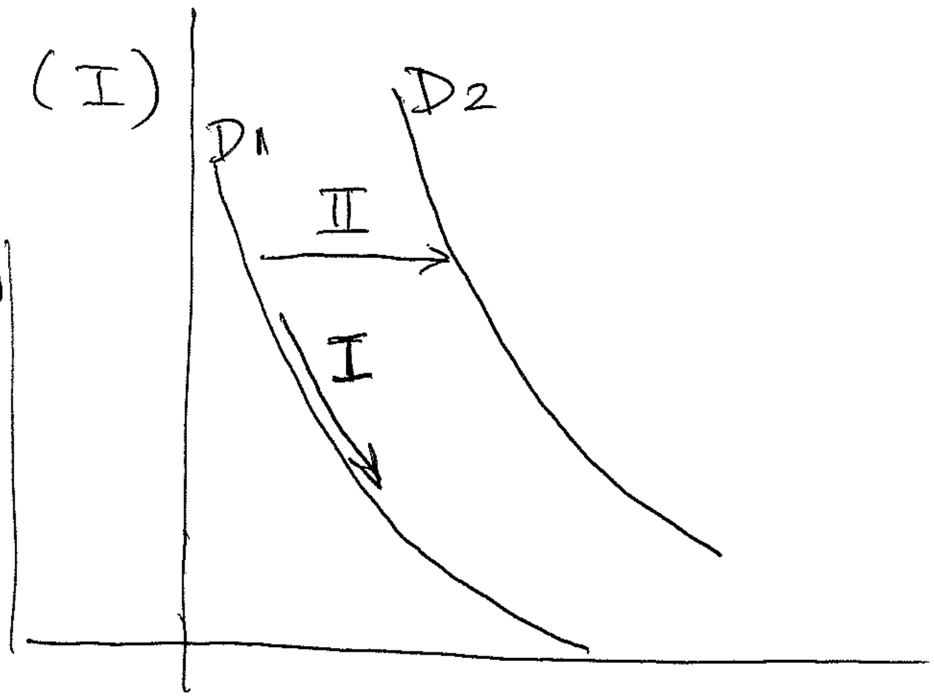
$20 \to 21$ $\quad \Delta p = 1$.

$$dq = \underbrace{q'(p)}_{-20}\,\underbrace{dp}_{1} = -20.$$

$$q(21) = q(20) + \Delta q \approx \underline{q(20)} + dq = 100 - 20 = 80.$$

Increase in quantity demanded (I)

vs. Increase in demand (II)



price elasticity of demand (PED)

we study.

$\boxed{\text{change in } q}$ in response to

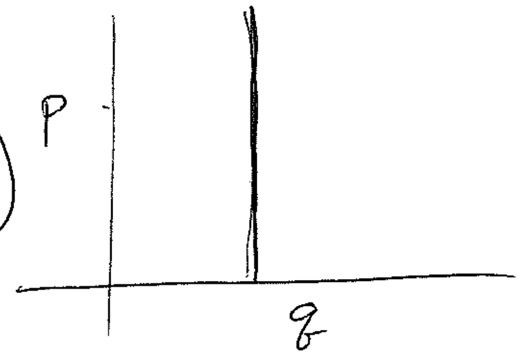
$\boxed{\text{change in } p}$.

We look at $\boxed{\text{relative changes}}$

$$PED = \frac{\frac{dq}{q}}{\frac{dp}{p}} = \frac{dq}{dp} \cdot \frac{p}{q}.$$

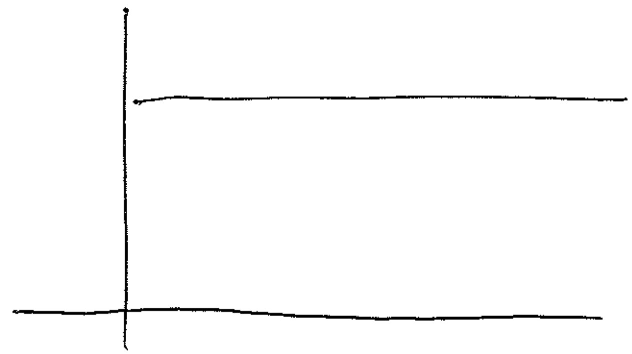
PED = 0, perfectly in elastic

(e.g. demand for salt over the entire market)



PED = ∞, perfectly elastic

(e.g. demand for corn for a particular farm)



$$PED = \frac{dq}{dp} \cdot \frac{p}{q}$$

$$q = b\left(1 - \frac{p}{a}\right)$$

$$\frac{dq}{dp} = -\frac{b}{a}$$

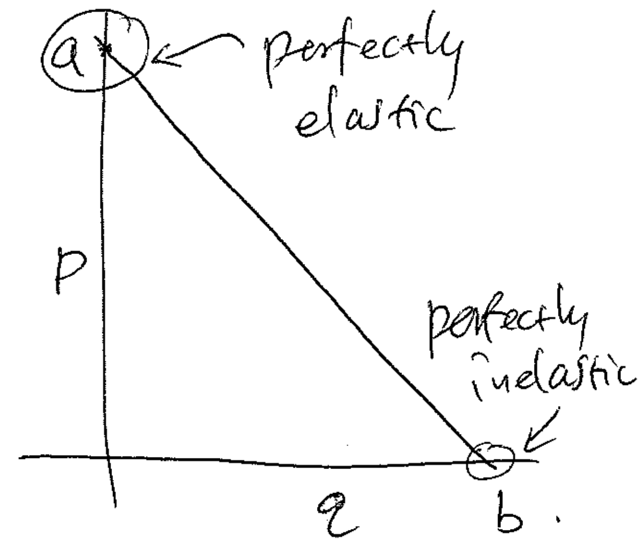
$$PED = -\frac{b}{a} \cdot \frac{p}{q}$$

near $q = b$, $p = 0$.

$$PED \approx 0$$

near $q = 0$, $p = a$

$$PED \approx \infty$$



$$\frac{p}{a} + \frac{q}{b} = 1$$

It does not imply

$PED = \text{const}$

§14.2 The indefinite integral

An antiderivative of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x)$$

Ex. (x^3+c) is an antiderivative of $3x^2$ for any C

Notation of indefinite integral

$$\underbrace{\int f(x)\,dx}_{\text{The general antiderivative of } f(x)} = F(x) + C .$$

Ex. $\int 3x^2\,dx = x^3 + C$

properties of $\int f(x)\,dx$

*). $\frac{d}{dx}\int f(x)\,dx = f(x)$

*). $\int \left(\frac{d}{dx}F(x)\right) dx = F(x)+C$

*). $\int \left(f_1(x)+f_2(x)\right) dx = \int f_1(x)\,dx + \int f_2(x)\,dx$

*) $\int k f(x)\,dx = k\int f(x)\,dx$

Integration Formulas.

*) $(x)' = 1 \longrightarrow \int 1\,dx = x+C$

$\boxed{\int k\,dx = kx+C}$

*) $\left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = x^{\alpha} \longrightarrow \boxed{\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1}+C}$

$(\alpha \neq -1)$ $\qquad$ $\alpha \neq -1$

*) $(\ln x)' = \frac{1}{x} \longrightarrow \int \frac{1}{x} dx = \ln x + C$

*) $(e^x)' = e^x \longrightarrow \int e^x dx = e^x + C$

Ex. $\int -\frac{1}{4} e^t dt = -\frac{1}{4} \int e^t dt = -\frac{1}{4} e^t + C$

Ex. $\int 3 \frac{1}{\sqrt{t}} dt = 3 \int t^{-\frac{1}{2}} dt = 3 \cdot \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$$= 3 \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = 6\sqrt{t} + C$$

Ex. $\int (2\sqrt{x} - 5e^x + \frac{1}{x}) dx = 2\int \sqrt{x}\, dx - 5\int e^x dx + \int \frac{1}{x} dx$

$$= 2 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 5e^x + \ln x + C = \frac{4}{3} x^{\frac{3}{2}} - 5e^x + \ln x + C$$

Ex $\int \frac{(x-1)(x+2)}{x^2}\,dx$

$(x-1)(x+2)$

$= x^2 - x + 2x - 2$

$= \boxed{x^2 + x - 2}$

$= \int \left(1 + \frac{1}{x} - 2\frac{1}{x^2}\right) dx$

$= x + \ln x - 2\frac{x^{-2+1}}{-2+1} + C$

$= x + \ln x + 2\frac{1}{x} + C.$