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Minimizing the maximum curvature of quadratic Bézier curves with a tetragonal concave polygonal boundary constraint

Ji-wung Choi [a,*], Renwick E. Curry [b], Gabriel H. Elkaim [b]

[a] Department of Intelligent Hydraulics and Automation, Tampere University of Technology, Tampere 33101, Finland

[b] Autonomous Systems Laboratory, Department of Computer Engineering, University of California, Santa Cruz, 95064, USA

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ABSTRACT

In curve design such as highway design and motion planning of autonomous vehicles, it may be important to minimize the maximum curvature. In this paper we address the problem of minimizing the maximum curvature of a quadratic Bézier curve within a boundary constraint determined by a tetragonal concave polygon. The curve is parameterized by lengths between its control points, called the "control lengths". Finally, numerical results demonstrate applicability of the method to smooth a piecewise linear path resulting from a path search technique. The results apply whenever it is desired to have a smooth transition between intersecting straight lines.

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1. Introduction

Bézier curves were invented in 1962 by the French engineer Pierre Bézier for designing automobile bodies. Today Bézier curves are widely used to interpolate a given set of data in computer aided geometric design. Controlling curvature is one of the most important requisites for the construction of the curves [1]. Controlling the curvature is not only related to "visual pleasantness", but also to the design, the kinematic, and the dynamic constraints in practical applications such as highway design [2], motion planning of autonomous vehicles [3–6], and boat or canoe building [7]. In the first two examples, since curvature tends to have an inverse correlation with speed of the vehicles [2], it may be desirable to minimize maximum curvature on a curve.

In the past, many techniques to control the curvature of Bézier curves have been discussed in the literature. In 1992, Sapidis and Frey first solved the problem of computing the maximum curvature of quadratic Bézier curves [1]. The maximum value is formulated by interpreting geometry of the control points and the hodograph of the curves. Deddi et al. transformed the formula in terms of lengths and areas determined by the control points [8]. They solved the problem of controlling the curvature along the whole length of a quadratic Bézier curve interpolating a given set of data by using the formula. In contrast to a quadratic curve, it is complex to evaluate the maximum curvature of a higher degree Bézier curve due to its high degree polynomial. Alternatively, spiral segments which have no interior curvature extrema [9] or piecewise quadratic curves to approximate higher degree curves [5] have been used.

In this paper we solve the problem of minimizing the maximum magnitude of curvature of a quadratic Bézier curve, in which the curve has two degrees of freedom: lengths between its control points, called "control lengths". Given upper limits and the angle between the two lengths, we present how to compute the optimal control lengths to minimize the maximum curvature of the curve determined by them. One of the properties of Bézier curves, the "convex hull property", means the resulting curve lies within the convex hull determined by its control points. We will extend the problem of minimizing the maximum curvature such that the curve lies within a tetragonal concave polygon determined by the control points of the curve and an additional point within the convex hull of the control points.

This is an important problem in motion planning of robots. Many path planning techniques discussed in the literature produce a piecewise linear path [10–13]. To satisfy kinematic feasibility of the robot to follow the path, it is necessary to make the path smooth. When the Bézier curve is used to interpolate the junction nodes of the path, there is no guarantee that the path will avoid obstacles [4]. A possible solution is to make a piecewise smooth path using a Bézier curve. Section 5 shows how the analysis on the optimal control lengths is used to smooth the linear paths.

This paper is organized as follows. Section 2 begins by describing the problems considered. The solutions to the problems are provided in Sections 3 and 4. Section 3 presents the optimal control lengths to minimize the maximum curvature of the quadratic

* Corresponding author. Tel.: +358 40 1981227.
E-mail addresses: ji.choi@tut.fi, jiwung@gmail.com (J.-w. Choi), rcurry@ucsc.edu (R.E. Curry), elkaim@soe.ucsc.edu (G.H. Elkaim).
URLs: http://users.soe.ucsc.edu/~jwchoi/ (J.-w. Choi), http://users.soe.ucsc.edu/~elkaim/ (G.H. Elkaim).

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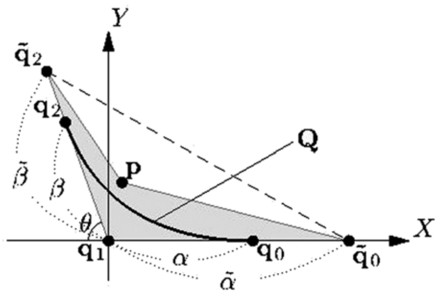


Fig. 1. Given $\mathbf{q}_1$, $\tilde{\mathbf{q}}_0$ and $\tilde{\mathbf{q}}_2$ that bounds $\mathbf{q}_0$ and $\mathbf{q}_2$ on $\mathbf{q}_1\tilde{\mathbf{q}}_0$ and $\mathbf{q}_1\tilde{\mathbf{q}}_2$, a quadratic Bézier curve $\mathbf{Q}$ illustrated by bold solid line is determined by θ, α and β. θ is the heading difference between $\tilde{\mathbf{q}}_2 - \mathbf{q}_1$ and $\mathbf{q}_1 - \tilde{\mathbf{q}}_0$. α and β denote $\|\mathbf{q}_0 - \mathbf{q}_1\|$ and $\|\mathbf{q}_2 - \mathbf{q}_1\|$, respectively. $\mathbf{p}$ defines the tetragonal concave $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$ within which $\mathbf{Q}$ must lie.

Bézier curve within a given triangle. Based on the analysis in the section, Section 4 solves the maximum curvature minimization problem with a tighter boundary constraint imposed by a tetragonal concave polygon. Section 5 demonstrates applicability of the solutions to practical applications such as path smoothing and computer design. Finally, Section 6 provides conclusions.

2. Problem statement

A quadratic Bézier curve $\mathbf{Q}(\lambda)$ is constructed by three control points $\mathbf{q}_0$, $\mathbf{q}_1$, and $\mathbf{q}_2$ such as

$$\mathbf{Q}(\lambda) = (1-\lambda)^2\mathbf{q}_0 + 2\lambda(1-\lambda)\mathbf{q}_1 + \lambda^2\mathbf{q}_2, \quad \lambda \in [0, 1]. \tag{1}$$

This paper addresses two problems to minimize the maximum curvature of a quadratic Bézier curve as follows.

1. Given a triangle $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$, find the points $\mathbf{q}_0$ and $\mathbf{q}_2$ on the segments $\mathbf{q}_1\tilde{\mathbf{q}}_0$ and $\mathbf{q}_1\tilde{\mathbf{q}}_2$, respectively, that minimize the maximum curvature of the quadratic Bézier curve with control points $\mathbf{q}_0$, $\mathbf{q}_1$, and $\mathbf{q}_2$.
2. The same problem with the additional constraint that the resulting Bézier curve lies anywhere within the quadrilateral $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$ where $\mathbf{p}$ is a given point that lies in the triangle $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$ (see Fig. 1).

2.1. Analytic interpretation

We transform the original Bézier curve, applying a set of affine maps, so as its control points become

$$\mathbf{q}_0 = (\alpha, 0), \qquad \mathbf{q}_1 = (0, 0), \qquad \mathbf{q}_2 = (-\beta\cos\theta, \beta\sin\theta), \tag{2}$$

where

$$\theta \in (0, \pi), \qquad \alpha \in (0, \tilde{\alpha}], \qquad \beta \in (0, \tilde{\beta}]. \tag{3}$$

The upper limits on control lengths are determined by segments of $\triangle\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$: $\tilde{\alpha} = \|\tilde{\mathbf{q}}_0 - \mathbf{q}_1\|$ and $\tilde{\beta} = \|\tilde{\mathbf{q}}_2 - \mathbf{q}_1\|$. In view of (2), $\mathbf{Q}(\lambda) = (x(\lambda), y(\lambda))^T$ can be written as

$$\begin{aligned} x(\lambda) &= (\alpha - \beta\cos\theta)\lambda^2 - 2\alpha\lambda + \alpha, \\ y(\lambda) &= \beta\sin\theta\lambda^2. \end{aligned} \tag{4}$$

Note that θ, $\tilde{\alpha}$, and $\tilde{\beta}$ are initially given by $\triangle\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$ so that $\mathbf{Q}(\lambda)$ has two degrees of freedom, α and β as represented in (4). Subsequently, the curvature of $\mathbf{Q}(\lambda)$ is given by

$$\kappa(\lambda; \alpha, \beta) = \frac{\dot{x}(\lambda)\ddot{y}(\lambda) - \dot{y}(\lambda)\ddot{x}(\lambda)}{(\dot{x}^2(\lambda) + \dot{y}^2(\lambda))^{\frac{3}{2}}} = \frac{\alpha\beta\sin\theta}{2[f(\lambda)]^{\frac{3}{2}}}, \tag{5}$$

where

$$f(\lambda) = (\alpha^2 + \beta^2 - 2\alpha\beta\cos\theta)\lambda^2 - 2\alpha(\alpha - \beta\cos\theta)\lambda + \alpha^2.$$

Let $\kappa_{\max}(\alpha, \beta)$ denote the maximum curvature of $\mathbf{Q}$ determined by α and β:

$$\kappa_{\max}(\alpha, \beta) = \max_{\lambda\in[0,1]} \kappa(\lambda; \alpha, \beta) = \frac{\alpha\beta\sin\theta}{2}[\min_{\lambda\in[0,1]} f(\lambda)]^{-\frac{3}{2}}. \tag{6}$$

With the definition of the variables, the problems that we consider are defined as follows.

Problem 1. Given θ, $\tilde{\alpha}$, and $\tilde{\beta}$,

minimize $\kappa_{\max}(\alpha, \beta)$
subject to
$0 < \alpha \le \tilde{\alpha}$,
$0 < \beta \le \tilde{\beta}$.

Problem 2. Given θ, $\tilde{\alpha}$, $\tilde{\beta}$, and $\mathbf{p}$ that lies inside of $\triangle\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$,

minimize $\kappa_{\max}(\alpha, \beta)$
subject to
$0 < \alpha \le \tilde{\alpha}$,
$0 < \beta \le \tilde{\beta}$,
$\mathbf{Q}(\alpha, \beta) \in \tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$.

We denote (α^*, β^*) the solutions of the problems. While the resulting curve, determined by the solution to Problem 1, lies within $\triangle\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$ by the convex hull property, that of Problem 2 lies within the tighter boundary constraint, $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$. The solutions to Problems 1 and 2 are presented in Sections 3 and 4, respectively.

3. Minimizing the maximum curvature of a quadratic Bézier curve

3.1. Maximum curvature of a quadratic Bézier curve

In view of (5), the behavior of $\kappa(\lambda; \alpha, \beta)$ depends on the behavior of $f(\lambda)$. The parabola $f(\lambda)$ is turning upwards since $\alpha^2 + \beta^2 - 2\alpha\beta\cos\theta = (\alpha-\beta)^2 + 2\alpha\beta(1-\cos\theta)$ is positive. Thus, $f(\lambda)$ has one global minimum at

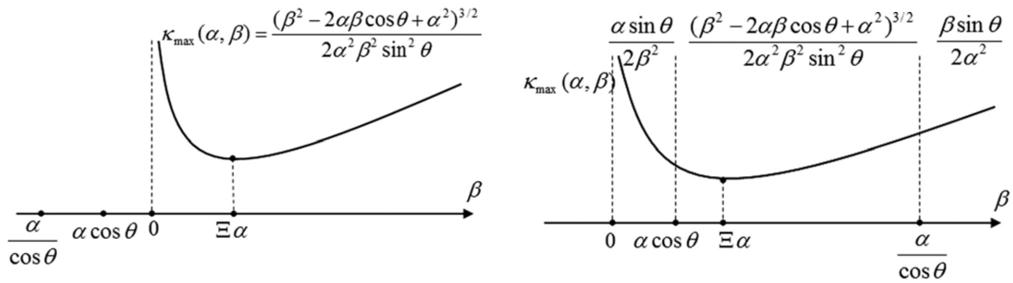
$$\lambda_0 = \frac{\alpha^2 - \alpha\beta\cos\theta}{\alpha^2 + \beta^2 - 2\alpha\beta\cos\theta}. \tag{7}$$

So, if $0 < \lambda_0 < 1$, then the maximum of $\kappa(\lambda; \alpha, \beta)$ is at the λ_0. Otherwise, when $\lambda_0 \le 0$ or $\lambda_0 \ge 1$, $\kappa(\lambda; \alpha, \beta)$ monotonically decreases or increases, respectively, as λ varies from 0 to 1, and thus its maximum is at 0 or 1. Thus,

$$\begin{aligned} &\text{If } \lambda_0 \le 0 \text{ or } \frac{\alpha}{\beta} \le \cos\theta, \text{ then } \kappa_{\max}(\alpha, \beta) = \kappa(0; \alpha, \beta) = \frac{\beta\sin\theta}{2\alpha^2}. \\ &\text{If } \lambda_0 \ge 1 \text{ or } \frac{\beta}{\alpha} \le \cos\theta, \text{ then } \kappa_{\max}(\alpha, \beta) = \kappa(1; \alpha, \beta) = \frac{\alpha\sin\theta}{2\beta^2}. \\ &\text{If } 0 < \lambda_0 < 1 \text{ or } \frac{\alpha}{\beta} > \cos\theta \text{ and } \frac{\beta}{\alpha} > \cos\theta, \\ &\quad \text{then } \kappa_{\max}(\alpha, \beta) = \kappa(\lambda_0; \alpha, \beta) = \frac{(\beta^2 - 2\alpha\beta\cos\theta + \alpha^2)^{\frac{3}{2}}}{2\alpha^2\beta^2\sin^2\theta}. \end{aligned} \tag{8}$$

3.2. Minimization with respect to one control length

In this subsection we minimize $\kappa_{\max}(\alpha, \beta)$ with respect to either of α or β. Without loss of generality, let us find $\beta^* \in (0, \tilde{\beta}]$ that minimizes $\kappa_{\max}(\alpha, \beta)$ for a fixed α.



(a) $\theta \geq \pi/2$. (b) $\theta < \pi/2$.

Fig. 2. By Eq. (8) and Lemma 1, the shape of $\kappa_{\max}(\alpha, \beta)$ with respect to β is concave upward with minimum at $\beta = \Xi\alpha$.

Lemma 1. *If α is fixed, the function $\kappa_{\max}(\alpha, \beta)$ given by*

$$\kappa_{\max}(\alpha, \beta) = \frac{(\beta^2 - 2\alpha\beta\cos\theta + \alpha^2)^{\frac{3}{2}}}{2\alpha^2\beta^2\sin^2\theta}, \tag{9}$$

has a unique global minimum at $\beta = \Xi\alpha$, where Ξ is given by

$$\Xi = \frac{-\cos\theta + \sqrt{\cos^2\theta + 8}}{2}. \tag{10}$$

Proof. Differentiating $\kappa_{\max}(\alpha, \beta)$ with respect to β yields

$$\frac{\partial}{\partial\beta}\kappa_{\max}(\alpha, \beta) = \frac{\sqrt{\beta^2 - 2\alpha\beta\cos\theta + \alpha^2}(\beta^2 + \alpha\beta\cos\theta - 2\alpha^2)}{2\alpha^2\beta^3\sin^2\theta},$$

so that

$$\frac{\partial}{\partial\beta}\kappa_{\max}(\alpha, \beta)\begin{cases} < 0, & \beta \in (0, \Xi\alpha), \\ = 0, & \beta = \Xi\alpha, \\ > 0, & \beta \in (\Xi\alpha, \infty). \end{cases}$$

So, $\kappa_{\max}(\alpha, \beta)$ given by (9) with respect to $\beta \in (0, \infty)$ is convex and appears to have only one minimum at $\beta = \Xi\alpha$. □

Based on the properties of $\kappa_{\max}(\alpha, \beta)$ with respect to β, the following theorem shows that the solution is at the minimum of $\tilde{\beta}$ and $\Xi\alpha$.

Theorem 1. *If α is fixed, $\beta^* \in (0, \tilde{\beta}]$ that minimizes $\kappa_{\max}(\alpha, \beta)$ is*

$$\beta^* = \min\left(\tilde{\beta}, \Xi\alpha\right). \tag{11}$$

Proof. Eq. (8) and Lemma 1 show that $\kappa_{\max}(\alpha, \beta)$ is a continuous function of β, that it is concave upward, and that it has a minimum at $\Xi\alpha$. See Fig. 2. Thus $\kappa_{\max}(\alpha, \beta)$ decreases as β increases from zero to $\Xi\alpha$, so $\kappa_{\max}(\alpha, \beta)$ is minimized at $\beta = \tilde{\beta}$ for $\tilde{\beta} < \Xi\alpha$. □

It is straightforward to see that Lemma 1 and Theorem 1 still hold true after exchanging α and β.

3.3. Optimal control lengths

Here we use the results of the previous subsection to find α^* and β^* that solve Problem 1.

Theorem 2. *The solution of the Problem 1 is given by*

$$\alpha^* = \min\left(\tilde{\alpha}, \Xi\tilde{\beta}\right), \qquad \beta^* = \min\left(\tilde{\beta}, \Xi\tilde{\alpha}\right). \tag{12}$$

Proof. If we introduce $\xi = \beta/\alpha$, then $\kappa_{\max}(\alpha, \beta)$ is rewritten from (8) to

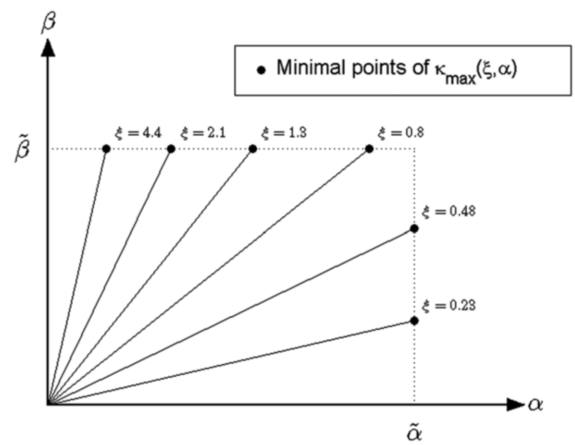


Fig. 3. Given $\xi = \beta/\alpha$, $\kappa_{\max}(\xi, \alpha)$ takes its minimum on the edge $\alpha = \tilde{\alpha}$ or $\beta = \tilde{\beta}$.

$$\kappa_{\max}(\xi, \alpha) = \begin{cases} \dfrac{\xi\sin\theta}{2}\cdot\dfrac{1}{\alpha}, & \text{if } \dfrac{1}{\xi} \leq \cos\theta, \\ \dfrac{\sin\theta}{2\xi^2}\cdot\dfrac{1}{\alpha}, & \text{if } \xi \leq \cos\theta, \\ \dfrac{(\xi^2 + 1 - 2\xi\cos\theta)^{3/2}}{(2\xi^2\sin^2\theta)}\cdot\dfrac{1}{\alpha}, & \text{if } \dfrac{1}{\xi} > \cos\theta \text{ and } \xi > \cos\theta. \end{cases}$$

Now it is a function of separated variables. As α increases $\kappa_{\max}(\xi, \alpha)$ decreases independently of ξ. So, given ξ, $\kappa_{\max}(\alpha, \beta)$ takes its minimum at $\alpha = \tilde{\alpha}$ or $\beta = \tilde{\beta}$ (see Fig. 3). Hence the global minimum of $\kappa_{\max}(\alpha, \beta)$ is

$$\kappa_{\max}(\alpha^*, \beta^*) = \min\left[\min_{\alpha\in(0,\tilde{\alpha}]}\kappa_{\max}(\alpha, \tilde{\beta}), \min_{\beta\in(0,\tilde{\beta}]}\kappa_{\max}(\tilde{\alpha}, \beta)\right], \tag{13}$$

where α and β that minimize $\kappa_{\max}(\alpha, \tilde{\beta})$ and $\kappa_{\max}(\tilde{\alpha}, \beta)$ are given by Theorem 1:

$$\begin{aligned} &\arg\min_{\alpha\in(0,\tilde{\alpha}]}\kappa_{\max}(\alpha, \tilde{\beta}) = \min\left(\tilde{\alpha}, \Xi\tilde{\beta}\right), \\ &\arg\min_{\beta\in(0,\tilde{\beta}]}\kappa_{\max}(\tilde{\alpha}, \beta) = \min\left(\tilde{\beta}, \Xi\tilde{\alpha}\right). \end{aligned} \tag{14}$$

Since (14) is expressed depending on the condition of $\tilde{\alpha}$ and $\tilde{\beta}$ compared to $\Xi\tilde{\beta}$ and $\Xi\tilde{\alpha}$, (α^*, β^*) of (13) is determined by each condition as follows.

(i) $\tilde{\alpha} < \Xi\tilde{\beta}$ and $\tilde{\beta} < \Xi\tilde{\alpha}$:

$$\kappa_{\max}(\alpha^*, \beta^*) = \min\left[\kappa_{\max}(\tilde{\alpha}, \tilde{\beta}), \kappa_{\max}(\tilde{\alpha}, \tilde{\beta})\right] = \kappa_{\max}(\tilde{\alpha}, \tilde{\beta})$$

(ii) $\tilde{\alpha} < \Xi\tilde{\beta}$ and $\tilde{\beta} \geq \Xi\tilde{\alpha}$:

$$\kappa_{\max}(\alpha^*, \beta^*) = \min\left[\kappa_{\max}(\tilde{\alpha}, \tilde{\beta}), \kappa_{\max}(\tilde{\alpha}, \Xi\tilde{\alpha})\right] = \kappa_{\max}(\tilde{\alpha}, \Xi\tilde{\alpha})$$

(iii) $\tilde{\alpha} \geq \Xi\tilde{\beta}$ and $\tilde{\beta} < \Xi\tilde{\alpha}$:

$$\kappa_{\max}(\alpha^*, \beta^*) = \min\left[\kappa_{\max}(\Xi\tilde{\beta}, \tilde{\beta}), \kappa_{\max}(\tilde{\alpha}, \tilde{\beta})\right] = \kappa_{\max}(\Xi\tilde{\beta}, \tilde{\beta})$$

(iv) $\tilde{\alpha} \geq \Xi\tilde{\beta}$ and $\tilde{\beta} \geq \Xi\tilde{\alpha}$:
Since $\Xi > 0$ and $\Xi^2 > 1$,

$$\tilde{\alpha} \geq \Xi\tilde{\beta} \Rightarrow \Xi\tilde{\alpha} \geq \Xi^2\tilde{\beta} > \tilde{\beta},$$

so the condition, $\tilde{\alpha} \geq \Xi\tilde{\beta}$ and $\tilde{\beta} \geq \Xi\tilde{\alpha}$, does not arise.

Incorporating (i)–(iii) yields (12). □

4. Minimizing maximum curvature of quadratic Bézier curves with a boundary constraint

In this section, we solve Problem 2 based on the analysis in the previous section. This section consists of three subsections. Section 4.1 presents the condition of $(\tilde{\alpha}, \tilde{\beta}, \theta)$ such that the burden of the boundary constraint of the problem is eliminated for all $\alpha \in (0, \tilde{\alpha}]$ and all $\beta \in (0, \tilde{\beta}]$, thus the computation of (α^*, β^*) becomes as easy as that of Theorem 2 in Section 3. If the condition is not satisfied, then Problem 2 will be more challenging. Section 4.2 shows that when the condition is not met, Theorem 2 is used with the range of α which is narrowed down to satisfy the boundary constraint. Finally, Section 4.3 summarizes the solution and includes the worst case when no condition of Section 4.1 or Section 4.2 is satisfied.

4.1. Independency condition from the boundary constraint

We begin with the constraint imposed on $\mathbf{p} = (p_x, p_y)$. Since $\mathbf{p} = (p_x, p_y)$ is constrained to lie within $\triangle\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$, it must fulfill the following constraints:

$$\begin{aligned} &0 < p_y < \tilde{\beta}\sin\theta, \\ &p_y\cos\theta + p_x\sin\theta > 0, \qquad (15) \\ &p_y(\tilde{\alpha} + \tilde{\beta}\cos\theta) + p_x\tilde{\beta}\sin\theta - \tilde{\alpha}\tilde{\beta}\sin\theta < 0. \end{aligned}$$

Let us introduce notation to describe the theorem provided in this subsection. Given α, let $\beta_p(\alpha)$ denote the control length such that $\mathbf{Q}_p(\alpha)$, (the curve constructed by the α and $\beta_p(\alpha)$) passes through $\mathbf{p}$, as shown in Fig. 4(a). It is important to note that, for a given $\alpha \in (0, \tilde{\alpha}]$, if $\beta_p(\alpha) < \tilde{\beta}$, then the control length $\beta_p(\alpha)$ is the upper limit of β to satisfy the boundary constraint $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$. This is because each quadratic polynomial Bézier curve is a parabolic segment so that the entire curve determined by the α and $\beta \in (0, \beta_p(\alpha)]$ lies within the region enclosed by $\mathbf{q}_0\mathbf{q}_1$, $\mathbf{q}_1\mathbf{q}_2$, and $\mathbf{Q}_p(\alpha)$. On the contrary, the curve determined by the α and $\beta > \beta_p(\alpha)$, illustrated as the dashed curve in Fig. 4(a), violates the boundary constraint.

The quadratic Bézier curve determined by $\tilde{\alpha}$ and $\tilde{\beta}$ is denoted by $\tilde{\mathbf{Q}}$. Note that the all curves determined by $\alpha \in (0, \tilde{\alpha}]$ and $\beta \in (0, \tilde{\beta}]$ lie within the region enclosed by $\tilde{\mathbf{q}}_0\mathbf{q}_1$, $\mathbf{q}_1\tilde{\mathbf{q}}_2$, and $\tilde{\mathbf{Q}}$. So, if $\mathbf{p}$ is outside of the region as illustrated in Fig. 4(b), then all curves satisfy the boundary constraint. In this case, no quadratic curve passes through $\mathbf{p}$ so that $\beta_p(\tilde{\alpha}) > \tilde{\beta}$. This property is useful to simplify the solution to Problem 2 as presented in the Theorem 3.

Let $\lambda_p \in (0, 1)$ denote the curve parameter such that $\mathbf{Q}(\lambda_p) = \mathbf{p}$. Incorporating the definition with (4) yields

$$(\alpha - \beta_p(\alpha)\cos\theta)\lambda_p^2 - 2\alpha\lambda_p + \alpha = p_x, \qquad (16)$$

$$\beta_p(\alpha)\lambda_p^2\sin\theta = p_y. \qquad (17)$$

Solving (16) and (17),

$$\lambda_p = 1 - \sqrt{\frac{p_x + p_y\cot\theta}{\alpha}}. \qquad (18)$$

Substituting λ_p obtained above into (17) yields $\beta_p(\alpha)$:

$$\beta_p(\alpha) = \frac{K_\beta}{(1 - \sqrt{K_\alpha/\alpha})^2}, \qquad (19)$$

where K_β and K_α are given by

$$K_\beta = \frac{p_y}{\sin\theta}, \qquad K_\alpha = p_x + p_y\cot\theta. \qquad (20)$$

The length $\alpha_p(\beta)$ is defined by exchanging α and β. Since $\beta_p(\alpha_p(\beta)) = \beta$, $\alpha_p(\beta)$ is obtained by substituting $\alpha_p(\beta)$ for α in (19) and equating it to β:

$$\alpha_p(\beta) = \frac{K_\alpha}{(1 - \sqrt{K_\beta/\beta})^2}. \qquad (21)$$

Lemma 2. *The functions $\beta_p(\alpha)$ and $\alpha_p(\beta)$ have the following properties.*

(i) *They are decreasing functions of α and β, respectively.*
(ii) *$\alpha_p(\tilde{\beta}) < \tilde{\alpha}$ if and only if $\beta_p(\tilde{\alpha}) < \tilde{\beta}$.*

Proof.

(i) Without loss of generality, differentiating $\beta_p(\alpha)$ with respect to α yields

$$\frac{d\beta_p(\alpha)}{d\alpha} = -K_\beta K_\alpha^{\frac{1}{2}}\alpha^{-\frac{3}{2}}\lambda_p^{-3} < 0.$$

(ii) From the first property,

$$\alpha_p(\tilde{\beta}) < \tilde{\alpha} \Rightarrow \beta_p(\alpha_p(\tilde{\beta})) > \beta_p(\tilde{\alpha}) \Rightarrow \tilde{\beta} > \beta_p(\tilde{\alpha}),$$

$$\beta_p(\tilde{\alpha}) \geq \tilde{\beta} \Rightarrow \alpha_p(\beta_p(\tilde{\alpha})) \leq \alpha_p(\tilde{\beta}) \Rightarrow \tilde{\alpha} \geq \alpha_p(\tilde{\beta}). \quad \square$$

With the use of the notation, the following theorem provides the condition that the boundary constraint can be neglected so that α^* and β^* are obtained simply by using Theorem 2, the unconstrained solution to Problem 1.

Theorem 3. *If*

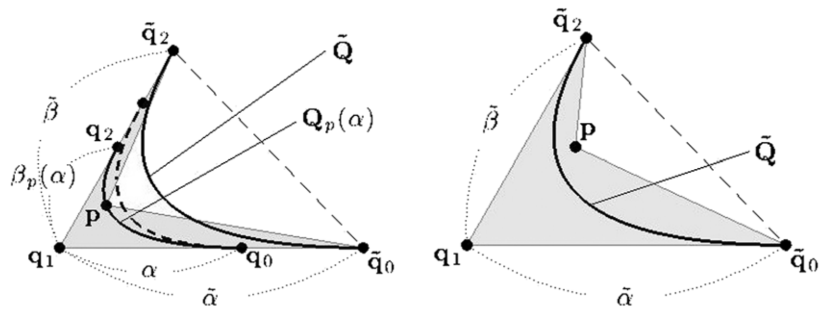
(i) *$\tilde{\beta} \leq \beta_p(\tilde{\alpha})$ or*
(ii) *$\Xi\tilde{\alpha} \leq \beta_p(\tilde{\alpha}) < \tilde{\beta}$ or*
(iii) *$\Xi\tilde{\beta} \leq \alpha_p(\tilde{\beta}) < \tilde{\alpha}$,*

then the solution to Problem 2 is

$$\alpha^* = \min\left(\tilde{\alpha}, \Xi\tilde{\beta}\right), \qquad \beta^* = \min\left(\tilde{\beta}, \Xi\tilde{\alpha}\right). \qquad (22)$$

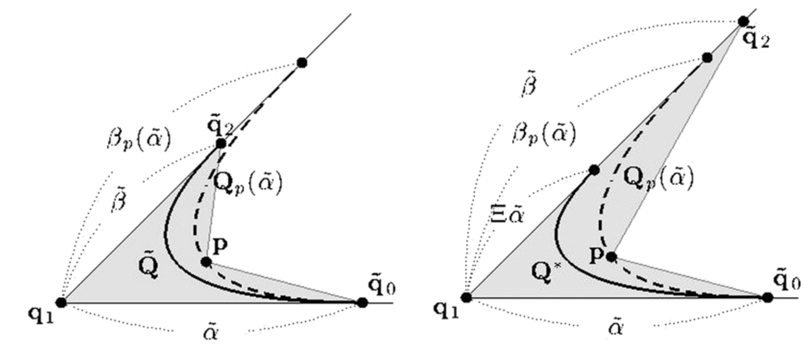
Proof.

(i) $\tilde{\beta} \leq \beta_p(\tilde{\alpha})$ (and thus $\tilde{\alpha} \leq \alpha_p(\tilde{\beta})$ by Lemma 2-(ii)): As illustrated in Fig. 5(a), if $\tilde{\beta} \leq \beta_p(\tilde{\alpha})$, $\tilde{\mathbf{Q}}$ lies within $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$. The entire curve constructed by $\alpha \in (0, \tilde{\alpha}]$ and $\beta \in (0, \tilde{\beta}]$ lies within the region enclosed by $\tilde{\mathbf{q}}_0\mathbf{q}_1$, $\mathbf{q}_1\tilde{\mathbf{q}}_2$ and $\tilde{\mathbf{Q}}$, and hence lies within $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$.
(ii) $\Xi\tilde{\alpha} \leq \beta_p(\tilde{\alpha}) < \tilde{\beta}$: Without considering the boundary constraint, if $\Xi\tilde{\alpha} < \tilde{\beta}$, then $(\alpha^*, \beta^*) = (\tilde{\alpha}, \Xi\tilde{\alpha})$ by Theorem 2. Let $\mathbf{Q}^*$ denote the curve constructed by the $(\alpha^*, \beta^*) = (\tilde{\alpha}, \Xi\tilde{\alpha})$. As illustrated in Fig. 5(b), if $\beta_p(\tilde{\alpha}) < \tilde{\beta}$, the curve



(a) When $\mathbf{p}$ is inside of $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\tilde{\mathbf{Q}}$. (b) When $\mathbf{p}$ is outside of $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\tilde{\mathbf{Q}}$.

Fig. 4. Given α, $\beta_p(\alpha)$ denotes the control length such that the curve constructed by the α and $\beta_p(\alpha)$ passes through $\mathbf{p}$. The curve determined by $\tilde{\alpha}$ and $\tilde{\beta}$ is denoted by $\tilde{\mathbf{Q}}$.



(a) Case (i). (b) Case (ii).

Fig. 5. The cases when it is not necessary to be concerned with the boundary constraint of Problem 2.

$\mathbf{Q}_p(\tilde{\alpha})$ lies within $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$. So, if $\Xi\tilde{\alpha} \le \beta_p(\tilde{\alpha})$, $\mathbf{Q}^*$ lies within the region enclosed by $\tilde{\mathbf{q}}_0\mathbf{q}_1$, $\mathbf{q}_1\tilde{\mathbf{q}}_2$ and $\mathbf{Q}_p(\tilde{\alpha})$, and hence lies within $\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2\mathbf{p}$.

(iii) $\Xi\tilde{\beta} \le \alpha_p(\tilde{\beta}) < \tilde{\alpha}$: Without loss of generality, exchanging α and β in (ii) holds.

Thus, if one of the conditions in (i)–(iii) holds, the solution to Problem 2 is same as that to Problem 1, given by (22). □

4.2. Narrowing down the range of control length

Theorem 3 provides a condition that the burden of the boundary constraint of Problem 2 is eliminated for all $\alpha \in (0, \tilde{\alpha}]$ and all $\beta \in (0, \tilde{\beta}]$. If the condition is not satisfied, then the solution to Problem 2 must be calculated taking into account the boundary constraint. Working analogously to Section 3.1, let us calculate $\beta^* \in (0, \tilde{\beta}]$ that minimizes $\kappa_{\max}(\alpha, \beta)$ when α is fixed, but under the boundary constraint. Then, $\alpha^* \in (0, \tilde{\alpha}]$ is determined by minimizing $\kappa_{\max}(\alpha, \beta^*)$ with the calculated β^* for all α.

The data $\tilde{\alpha}, \tilde{\beta}, \theta$ and $\mathbf{p}$ shown in Fig. 6(a) are such that no condition of Theorem 3 is satisfied. Without loss of generality, we will solve Problem 2 by decreasing α starting from $\tilde{\alpha}$. Recall that, given $\alpha \in [\alpha_p(\tilde{\beta}), \tilde{\alpha}]$, the range of β to satisfy the boundary constraint is $\beta \in (0, \beta_p(\alpha)]$. Fig. 6(b) shows that the upper limit $\beta_p(\alpha)$ monotonically increases from $\beta_p(\tilde{\alpha})$ to $\tilde{\beta}$ as α decreases from $\tilde{\alpha}$ to $\alpha_p(\tilde{\beta})$ (see Lemma 2-(i)). Once α reaches $\alpha_p(\tilde{\beta})$, the remaining $\alpha \in (0, \alpha_p(\tilde{\beta})]$ and all $\beta \in (0, \tilde{\beta}]$ satisfy the boundary constraint, as shown in Fig. 6(c). Thus for the remaining part, (α^*, β^*) are easily determined by using Theorem 2.

As pointed out, $\beta_p(\alpha)$ increases as α decreases by Lemma 2-(i). On the other hand, since $\Xi > 0$, $\Xi\alpha$ decreases as α decreases. So, as illustrated as dashed arrows in Fig. 6(b), $\Xi\alpha$ and $\beta_p(\alpha)$ will be equal at a certain α. We denote such α as α_c:

$$\beta_p(\alpha_c) = \Xi\alpha_c.$$

Equating (19) and $\Xi\alpha$ yields

$$\alpha_c = \left(\sqrt{K_\alpha} + \sqrt{\frac{K_\beta}{\Xi}}\right)^2. \tag{23}$$

Also, let us denote α_m such that $\beta_p(\alpha_m)/\alpha_m = \cos\theta$. So,

$$\alpha_m = \left(\sqrt{K_\alpha} + \sqrt{\frac{K_\beta}{|\cos\theta|}}\right)^2. \tag{24}$$

Referring to (8), α_m is the α where the expression of $\kappa_{\max}(\alpha, \beta_p(\alpha))$ changes form (see (26)). Note that $\alpha_m > \alpha_c$ because $\cos\theta < 1 < \Xi$.

Consider the problem of finding β^* for a given $\alpha \in (\alpha_c, \tilde{\alpha}]$ and all $\beta \in (0, \beta_p(\alpha)]$. Note that the upper limit of β is $\beta_p(\alpha)$ to satisfy the boundary constraint. Incorporating Theorem 1 and the definition of α_c yields

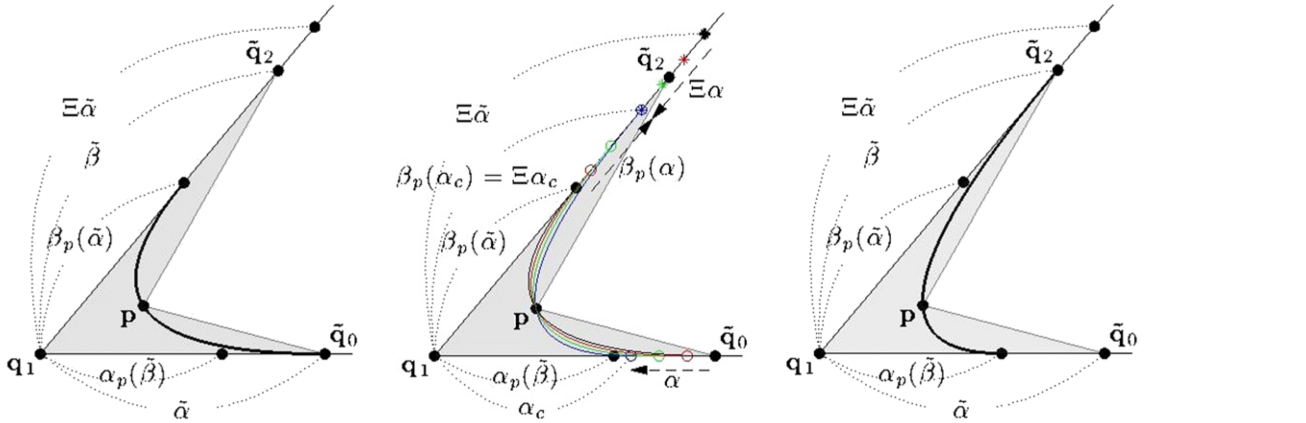
$$\beta^* = \min\left(\beta_p(\alpha), \Xi\alpha\right) = \beta_p(\alpha), \quad \alpha \in (\alpha_c, \tilde{\alpha}],$$
$$\forall\beta \in (0, \beta_p(\alpha)]. \tag{25}$$

If $\alpha_m < \tilde{\alpha}$, then using (8) and the above yields

$$\kappa_{\max}(\alpha, \beta^*) = \begin{cases} \dfrac{(\beta_p^2(\alpha) - 2\alpha\beta_p(\alpha)\cos\theta + \alpha^2)^{\frac{3}{2}}}{2\alpha^2\beta_p^2(\alpha)\sin^2\theta}, \\ \quad \alpha \in (\alpha_c, \alpha_m),\ \forall\beta \in (0, \beta_p(\alpha)], \\ \dfrac{\alpha\sin\theta}{2\beta_p^2(\alpha)}, \quad \alpha \in [\alpha_m, \tilde{\alpha}],\ \forall\beta \in (0, \beta_p(\alpha)]. \end{cases} \tag{26}$$

For $\alpha \in [\alpha_m, \tilde{\alpha}]$, differentiating $\kappa_{\max}(\alpha, \beta^*)$ with respect to α yields

$$\frac{\partial\kappa_{\max}}{\partial\alpha}(\alpha, \beta^*) = \frac{\sin\theta}{2\beta_p^2(\alpha)}\left(1 - 2\frac{\alpha}{\beta_p(\alpha)}\cdot\frac{d\beta_p(\alpha)}{d\alpha}\right) > 0,$$
$$\alpha \in [\alpha_m, \tilde{\alpha}], \tag{27}$$



(a) Configuration of $\beta_p(\tilde{\alpha})$ and $\Xi\tilde{\alpha}$. (b) Sliding α down from $\tilde{\alpha}$. (c) Terminal state when $\alpha = \alpha_p(\tilde{\beta})$.

Fig. 6. The geometry when no condition of Theorem 3 is met. $\Xi\alpha$ decreases but $\beta_p(\alpha)$ increases as α decreases from $\tilde{\alpha}$. Once α reaches $\alpha_p(\tilde{\beta})$, the remaining $\alpha \in (0, \alpha_p(\tilde{\beta})]$ and all $\beta \in (0, \tilde{\beta}]$ satisfy the boundary constraint.

because $d\beta_p(\alpha)/d\alpha < 0$ by Lemma 2-(i). That is, $\kappa_{\max}(\alpha, \beta^*)$ decreases as α decreases from $\tilde{\alpha}$ to α_m. So, if we let α_{m^+} be any α that is in $[\alpha_m, \tilde{\alpha})$, then $\kappa_{\max}(\alpha, \beta^*) = \kappa_{\max}(\alpha, \beta_p(\alpha))$ is minimized by taking the smallest possible α for all $\alpha \in [\alpha_{m^+}, \tilde{\alpha}]$ and all $\beta \in (0, \beta_p(\alpha)]$.

Now we derive the theorem that computes (α^*, β^*) by using Theorem 2 with the range of α, which is narrowed down to satisfy the boundary constraint.

Theorem 4. *If no condition of Theorem 3 is met and $\alpha_p(\tilde{\beta}) \geq \alpha_m$, then the solution of the Problem 2 is given by*

$$\alpha^* = \alpha_p(\tilde{\beta}), \qquad \beta^* = \tilde{\beta}. \tag{28}$$

Proof. Incorporating the hypothesis $\alpha_p(\tilde{\beta}) \geq \alpha_m$ with logical inversion of the conditions of Theorem 3 yields

$$\alpha_m \leq \alpha_p(\tilde{\beta}) < \tilde{\alpha}, \tag{29}$$

$$\alpha_m \leq \alpha_p(\tilde{\beta}) < \Xi\tilde{\beta}. \tag{30}$$

By (29), as pointed out in the above, $\kappa_{\max}(\alpha, \beta^*) = \kappa_{\max}(\alpha, \beta_p(\alpha))$ is minimized by taking the smallest possible α for all $\alpha \in [\alpha_p(\tilde{\beta}), \tilde{\alpha}]$ and all $\beta \in (0, \beta_p(\alpha)]$:

$$\begin{aligned} &\alpha^* = \alpha_p(\tilde{\beta}), \qquad \beta^* = \beta_p(\alpha_p(\tilde{\beta})) = \tilde{\beta}, \\ &\forall\alpha \in [\alpha_p(\tilde{\beta}), \tilde{\alpha}],\ \forall\beta \in (0, \beta_p(\alpha)]. \end{aligned} \tag{31}$$

Recall that the upper limit of β is $\beta_p(\alpha)$ to satisfy the boundary constraint. On the other hand, all $\alpha \in (0, \alpha_p(\tilde{\beta})]$ and all $\beta \in (0, \tilde{\beta}]$ satisfy the boundary constraint (See Fig. 6(c)). Note that $(\alpha^*, \beta^*) = (\alpha_p(\tilde{\beta}), \tilde{\beta})$ of (31) is the upper limit of the interval, $\alpha \in (0, \alpha_p(\tilde{\beta})]$ and $\beta \in (0, \tilde{\beta}]$. Thus the solution to Problem 2 is determined by using Theorem 2 for $\alpha \in (0, \alpha_p(\tilde{\beta})]$ and $\beta \in (0, \tilde{\beta}]$:

$$\alpha^* = \min(\alpha_p(\tilde{\beta}), \Xi\tilde{\beta}) = \alpha_p(\tilde{\beta}), \qquad \beta^* = \min(\tilde{\beta}, \Xi\alpha_p(\tilde{\beta})) = \tilde{\beta},$$

where $\alpha^* = \alpha_p(\tilde{\beta})$ is obtained by using (30). Note that $\alpha_c < \alpha_m \leq \alpha_p(\tilde{\beta})$ and $\beta_p(\alpha_c) = \Xi\alpha_c$. While $\beta_p(\alpha)$ is a decreasing function of α (Lemma 2-(i)), $\Xi\alpha$ is proportional to α. So,

$$\beta_p(\alpha_p(\tilde{\beta})) = \tilde{\beta} < \Xi\alpha_p(\tilde{\beta}),$$

and hence $\beta^* = \min(\tilde{\beta}, \Xi\alpha_p(\tilde{\beta})) = \tilde{\beta}$. □

4.3. Optimal control lengths

Finally, the following theorem summarizes the solution to Problem 2 by incorporating Theorems 3 and 4, and presenting the computation of (α^*, β^*) in the worst case in which no condition of the two theorems is satisfied.

Theorem 5. *The optimum values α^*, β^* for solving the Problem 2 are obtained as follows.*

If $\tilde{\beta} \leq \beta_p(\tilde{\alpha})$ or $\Xi\tilde{\alpha} \leq \beta_p(\tilde{\alpha}) < \tilde{\beta}$ or $\Xi\tilde{\beta} \leq \alpha_p(\tilde{\beta}) < \tilde{\alpha}$, then

$$\alpha^* = \min\left(\tilde{\alpha}, \Xi\tilde{\beta}\right), \qquad \beta^* = \min\left(\tilde{\beta}, \Xi\tilde{\alpha}\right). \tag{32}$$

Else if $\alpha_p(\tilde{\beta}) \geq \alpha_m$, then

$$\alpha^* = \alpha_p(\tilde{\beta}), \qquad \beta^* = \tilde{\beta}. \tag{33}$$

Else α^* and β^* are as given in Box I.

Proof. Since Eqs. (32) and (33) have been proved in Theorems 3 and 4, we only prove (34). Note that (34) is used if no condition of Theorem 3 is met and $\alpha_p(\tilde{\beta}) < \alpha_m$. As pointed out in the previous subsection, if $\alpha_m < \tilde{\alpha}$, $\kappa_{\max}(\alpha, \beta^*) = \kappa_{\max}(\alpha, \beta_p(\alpha))$ is minimized by taking the smallest possible α for all $\alpha \in [\alpha_m, \tilde{\alpha}]$ and all β that satisfies the boundary constraint:

$$\begin{aligned} &\alpha^* = \min(\alpha_m, \tilde{\alpha}), \qquad \beta^* = \beta_p(\alpha^*), \\ &\forall\alpha \in [\min(\alpha_m, \tilde{\alpha}), \tilde{\alpha}],\ \forall\beta \in (0, \tilde{\beta}]. \end{aligned} \tag{35}$$

Incorporating Theorem 2 with the definition of α_c yields

$$\begin{aligned} &\alpha^* = \alpha_c, \qquad \beta^* = \beta_p(\alpha_c) = \Xi\alpha_c, \\ &\forall\alpha \in (0, \alpha_c],\ \forall\beta \in (0, \tilde{\beta}]. \end{aligned} \tag{36}$$

From the proof of Theorem 4,

$$\begin{aligned} &\alpha^* = \alpha_p(\tilde{\beta}), \qquad \beta^* = \begin{cases} \beta_p(\alpha_p(\tilde{\beta})), & \text{if } \alpha_c < \alpha_p(\tilde{\beta}) \\ \Xi\alpha_p(\tilde{\beta}), & \text{if } \alpha_c \geq \alpha_p(\tilde{\beta}), \end{cases} \\ &\forall\alpha \in (0, \alpha_p(\tilde{\beta})],\ \forall\beta \in (0, \tilde{\beta}]. \end{aligned} \tag{37}$$

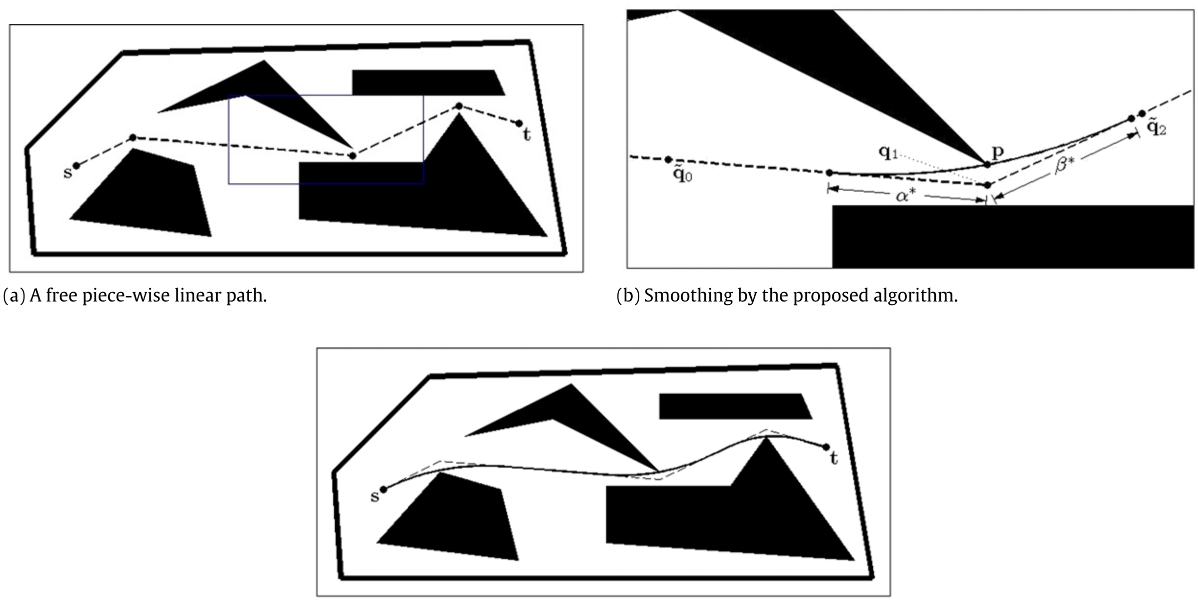
Incorporating (36) and (37) yields

$$\begin{aligned} &\alpha^* = \max(\alpha_c, \alpha_p(\tilde{\beta})), \qquad \beta^* = \beta_p(\alpha^*), \\ &\forall\alpha \in (0, \max(\alpha_c, \alpha_p(\tilde{\beta}))],\ \forall\beta \in (0, \tilde{\beta}]. \end{aligned} \tag{38}$$

$$\alpha^* = \arg\min \frac{\left((\sqrt{\alpha}-\sqrt{K_\alpha})^4 - 2K_\beta\cos\theta(\sqrt{\alpha}-\sqrt{K_\alpha})^2 + K_\beta^2\right)^{\frac{3}{2}}}{2K_\beta^2\sin^2\theta(\sqrt{\alpha}-\sqrt{K_\alpha})^2\alpha}, \quad \forall\alpha \in \left[\max(\alpha_c, \alpha_p(\tilde{\beta})), \min(\alpha_m, \tilde{\alpha})\right], \tag{34}$$

$$\beta^* = \beta_p(\alpha^*).$$

Box I.



(a) A free piece-wise linear path.

(b) Smoothing by the proposed algorithm.

(c) The resulting path.

Fig. 7. Path smoothing using the analysis presented here.

Note that Eq. (35) (and (37), respectively) represents that $\kappa_{\max}(\alpha, \beta^*) = \kappa_{\max}(\alpha, \beta_p(\alpha))$ is minimized by taking the lower (and upper) limit of α for all $\alpha \in [\min(\alpha_m, \tilde{\alpha}), \tilde{\alpha}]$ (and all $\alpha \in (0, \max(\alpha_c, \alpha_p(\tilde{\beta}))]$). Thus $\kappa_{\max}(\alpha^*, \beta^*)$ is solely determined by minimizing $\kappa_{\max}(\alpha, \beta^*) = \kappa_{\max}(\alpha, \beta_p(\alpha)), \forall\alpha \in \left[\max(\alpha_c, \alpha_p(\tilde{\beta})), \min(\alpha_m, \tilde{\alpha})\right]$. Incorporating Theorem 1 and (26) yields

$$\kappa_{\max}(\alpha, \beta^*) = \frac{(\beta_p^2(\alpha) - 2\alpha\beta_p(\alpha)\cos\theta + \alpha^2)^{\frac{3}{2}}}{2\alpha^2\beta_p^2(\alpha)\sin^2\theta},$$

$$\alpha \in \left[\max(\alpha_c, \alpha_p(\tilde{\beta})), \min(\alpha_m, \tilde{\alpha})\right].$$

Combining the above expression with (19) we are led to (34). □

In summary, recall that Problem 2 is the constrained optimization problem with two free variables α and β. Theorem 5 shows that the solution can be obtained analytically by (32) or (33) depending on the relationship of $\tilde{\alpha}$ and $\tilde{\beta}$. In the worst case, the problem can be reduced to the problem of finding the minimum of a function with a single variable α on a fixed interval Eq. (34).

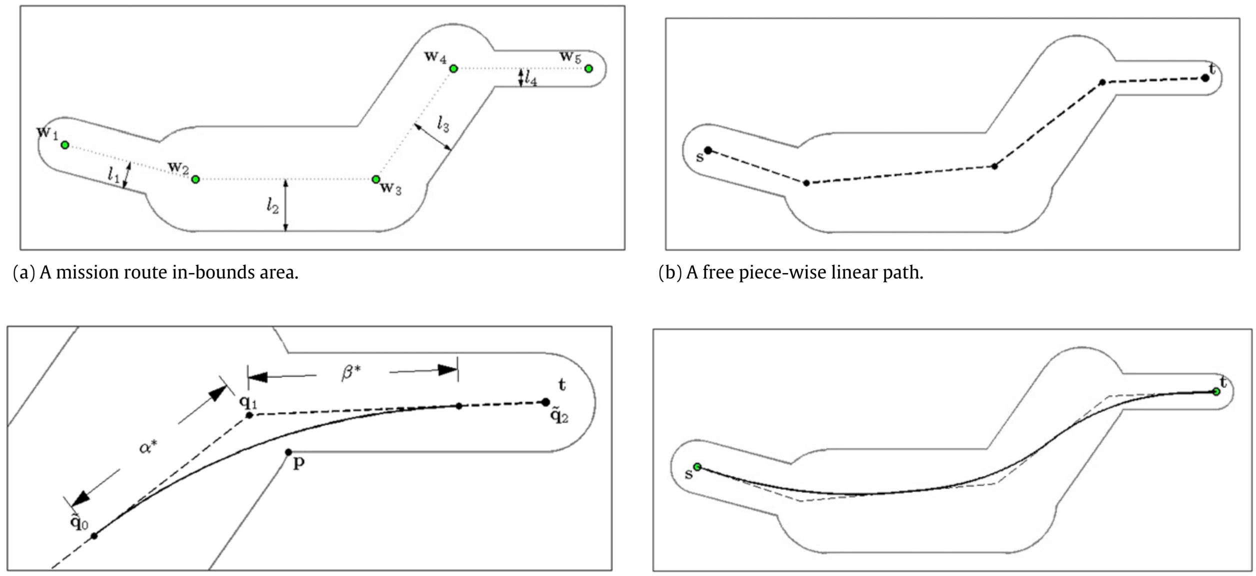
5. Application

In this section, we show that the analysis presented here can be usefully applied to practical applications such as path planning. In addition, these results can be used whenever a smooth transition is desired between two intersecting straight lines.

Path planning is one of the main problems in the field of robotics, especially navigation of autonomous vehicles. For vehicle viability, it is imperative to be able to generate safe paths in real time. Among many path planning methods literature, the work on search algorithms [10–13] leads to computationally efficient ways in discrete state spaces. However, the resulting paths are piecewise linear and not smooth, hence, do not satisfy kinematic feasibility constraints of the vehicle. Thus it is necessary to smooth out the piece-wise linear path so the vehicle may follow it.

Fig. 7 illustrates the path smoothing using the algorithm presented here. The path smoothing problem is formulated by inflating the obstacle radius to half the vehicle width, and that allows the path planning to assume that the vehicle is a single point. The free space is externally bounded by a polygon and internally bounded by four polygons and a free piece-wise linear path is given, as illustrated the dashed line (Fig. 7(a)). The start point and the target point are denoted as **s** and **t** in the figure. Quadratic Bézier curves are used to smooth sharp corners in the path. $\mathbf{q}_1$ is placed on a junction node. $\tilde{\mathbf{q}}_0$ and $\tilde{\mathbf{q}}_2$ are placed on the midpoints of line segments connecting two nodes except the first $\tilde{\mathbf{q}}_0$ placed on **s** and the last $\tilde{\mathbf{q}}_2$ on **t**. **p** is assigned as the tip of the polygon, which lies within $\triangle\tilde{\mathbf{q}}_0\mathbf{q}_1\tilde{\mathbf{q}}_2$. Each pair of control points $\mathbf{q}_0$ and $\mathbf{q}_2$ is determined by α^* and β^* calculated by applying Theorem 5 (Fig. 7(b)). Finally, the resulting path consists of the quadratic Bézier curves and the line segments connecting the curves (Fig. 7(c)).

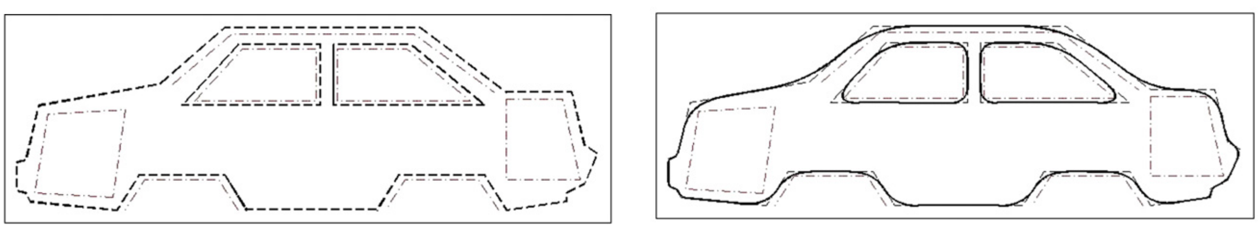
Another motion planning application is presented in Fig. 8. The path is planned for a route bounded by an area specified by a



(a) A mission route in-bounds area. (b) A free piece-wise linear path.

(c) Smoothing by the proposed algorithm. (d) The resulting path.

Fig. 8. Path smoothing for a route in-bounds area.



(a) A linear outline of a car. (b) The smoothed outline.

Fig. 9. Car design using the proposed smoothing method.

series of waypoints $\mathbf{w}_1, \ldots, \mathbf{w}_n$ and corridor widths $l_1, \ldots, l_{n-1}$ (Fig. 8(a)). The mission route has been used for the DARPA Grand Challenge 2005 and more detailed specification is provided in [14]. The proposed path smoothing method is applied to the free piece-wise linear path connecting $\mathbf{w}_1, \mathbf{w}_n$, and certain points around $\mathbf{w}_2, \ldots, \mathbf{w}_{n-1}$ (Fig. 8(b)). The placement of the series of $\tilde{\mathbf{q}}_0$, $\mathbf{q}_1$, and $\tilde{\mathbf{q}}_2$ is the same as the previous path planning problem. $\mathbf{p}$ is placed on one of the intersection points between outer boundaries of the route. (The detailed calculation is presented in [15].) For each set of the points, smoothing the path segment by the algorithm (Fig. 8(c)) results in the smoothed path (Fig. 8(d)).

In addition, these results can be generally used to smooth any polyline or polygon with constrained lines or region. Consider the polygon and the constrained lines as illustrated the dashed line and the dotted dash line, respectively, in Fig. 9(a). The proposed smoothing method generates the smoothed curve against the constrained lines, as shown in Fig. 9(b).

6. Conclusions

This paper presents a solution to the problem of minimizing the maximum curvature of a quadratic Bézier curve parameterized by distances between its control points, with a boundary constraint determined by a tetragonal concave polygon. The numerical results demonstrate applicability of the analysis presented here, including path smoothing problems.

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Ji-wung Choi was born in South Korea. He received the B.S. degree in electrical engineering from Yonsei University at Seoul, South Korea in 2006, and Ph.D. degree in computer engineering from University of California, Santa Cruz in 2010. Currently, he is Postdoctoral Research Fellow at Tampere University of Technology in Finland. At the beginning his research was oriented in distributed predictive controllers. Currently, the research interests is in motion planning and optimal control of autonomous robots.

Renwick E. Curry received degrees from Middlebury College and MIT. He has been a Guidance and Control Engineer at the Draper Laboratory (Apollo Project), Associate Professor of Aeronautics and Astronautics at MIT, and Research Scientist at NASA Ames Research Center. He is co-founder and president of AASI (www.aasi.com), and Adjunct Professor of Computer Engineering at the University of California, Santa Cruz. Dr. Curry has applied the theory of optimization and control to the design of real-world systems in flight path optimization and revenue management. He has participated in numerous government and professional panels and committees, and has published over 70 technical papers and reports.

Gabriel H. Elkaim received his B.S. degree in mechanical/aerospace engineering from Princeton University in 1990, the M.S. and Ph.D. degrees from Stanford University in aeronautics and astronautics, in 1995 and 2002, respectively. In 2003, he joined the faculty of the University of California, Santa Cruz, where he is an associate professor in the Computer Engineering Department. His research interests include control systems, sensor fusion, GPS, system identification, and autonomous vehicle systems. His research focuses on intelligent autonomous vehicles, with an emphasis on robust guidance, navigation, and control strategies. Specifically, he has founded the Autonomous Systems Lab at UC Santa Cruz, and is currently developing an autonomous wing-sailed marine surface vehicle and off-road autonomous ground vehicles.