

# **One-Day Short Course on Bayesian Modeling, Inference and Prediction**

**Preface, Tentative Syllabus/Outline,  
and 1: Background and Basics**

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# Preface

This material provides a thorough introduction to (or review of) **basic methodological ideas and applications** in **Bayesian modeling, inference and prediction**, based on a series of **case studies** and assuming no previous exposure to Bayesian ideas or methods.

The case studies are drawn from **medicine** (diagnostic screening for HIV; hospital-specific prediction of patient-level mortality rates; hospital length of stay for premature births; a randomized controlled trial of in-home geriatric assessment) and the **physical sciences** (measurement of physical constants), but the methods illustrated will apply to a broad range of subject areas in the **natural** and **social sciences**.

The course is intended mainly for **applied statisticians** and will focus on **methods** and **applications** rather than **theory**; an understanding of probability and statistics at the level typically required for a **Master's degree in statistics** will provide sufficient mathematical background for participants.

Extensive **details** required for carrying out the analyses are provided below, including **hardcopy** of a number of sessions with a frequently-used **statistical computing** package (R), a leading **symbolic computing** package (Maple), and one of the two most widely available packages for **fitting** Bayesian models (BUGS/WinBUGS).

# Tentative Syllabus/Outline

**8.00–9.00am:** Check-in and coffee.

**9.00–9.30am:** (1) **Background and basics.**

**Quantification of uncertainty:** classical, frequentist, and Bayesian definitions of probability. **Subjectivity** and **objectivity**. **Case study:** Diagnostic screening for HIV.

Sequential learning; **Bayes' Theorem**. **Inference** (science) and **decision-making** (policy and business).

Bayesian **decision theory**; **coherence**. **Maximization** of **expected utility**.

**9.30–11.00am:** (2) **Exchangeability and conjugate modeling.**

Probability as **quantification of uncertainty about observables**. **Binary** outcomes. **Case Study:** Hospital-specific prediction of patient-level mortality rates.

Review of **frequentist modeling** and **maximum-likelihood inference**.

**Exchangeability** as a Bayesian concept parallel to frequentist **independence**.

**Prior**, **posterior**, and **predictive** distributions.

**Inference** and **prediction**. **Coherence** and **calibration**.

**Conjugate analysis**. Comparison with **frequentist modeling**.

# Tentative Syllabus/Outline (continued)

**11.00–11:15am:** Coffee break.

**11.15–noon:** (3) **Integer-valued outcomes; Poisson modeling.**

Case Study: Hospital length of stay for birth of premature babies.

**noon–12.30pm:** (4) **Continuous outcomes; Gaussian modeling.**

**Multivariate** unknowns; **marginal** posterior distributions.  
Case Study: Measurement of physical constants (NB10).

The **exponential family**; **conjugate priors**.

**12.30–1.30pm:** Lunch break.

**1.30–3.30pm:**

(5) **Simulation-based computation.**

**IID** sampling; **rejection** sampling.

Introduction to **Markov chain Monte Carlo** (MCMC) methods: the **Metropolis-Hastings** algorithm and **Gibbs sampling**.

User-friendly **implementation** of Gibbs and Metropolis-Hastings sampling via BUGS and WinBUGS.

Case Study: the NB10 data revisited.

MCMC **implementation** strategies.

# Tentative Syllabus/Outline (continued)

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**3.30–3.45pm:** Coffee break.

**3.45–4.40pm:** (6) **Hierarchical models: formulation, selection, and diagnostics.**

Poisson **fixed-effects** modeling. Case study: a randomized controlled trial of in-home geriatric assessment.

**Additive** and **multiplicative** treatment effects.

**Expansion** of a simple model that does not satisfy all diagnostic checks, by **embedding** it in a richer class of models of which it's a special case.

**Random-effects Poisson regression.** Hierarchical modeling with **latent variables** as an approach to **mixture modeling.**

**4.40–4.45pm:** Get-up-and-move-around break.

**4.45pm–5.30pm:** (7) **Bayesian model specification.**

**Predictive** diagnostics. Model selection as a **decision** problem.

Bayesian **cross-validation** as an approach to diagnostics: comparing outcomes from omitted cases with their predictive distributions given the rest of the data. 3CV: **3-way cross-validation.**

The **log score** as a model-selection method, and its relationship to the **deviance information criterion** (DIC).  
Case study: continuation of IHGA example.

# 1: Background and Basics

**1.1 Quantification of uncertainty:** Classical, frequentist and Bayesian definitions of probability

*Case study:* Diagnostic screening for HIV

Widespread **screening for HIV** has been proposed by some people in some countries (e.g., the U.S.).

Two blood tests that screen for HIV are widely available: *ELISA*, which is relatively **inexpensive** (roughly \$20) and fairly accurate; and *Western Blot (WB)*, which is **considerably more accurate but costs quite a bit more** (about \$100).

A new patient comes to **You**, a physician, with symptoms that suggest he may be HIV positive (Good, 1950: You = a generic person making uncertainty assessments).

## Questions

- Is it appropriate to use the language of **probability** to quantify Your uncertainty about the proposition  $A = \{\text{this patient is HIV positive}\}$ ?
- If so, **what kinds of probability** are appropriate, and how would You assess  $P(A)$  in each case?
- What strategy (e.g., *ELISA*, *WB*, both?) should You employ to **decrease Your uncertainty** about  $A$ ? If You decide to run a screening test, how should Your uncertainty be **updated** in light of the test results?

# The Meaning of Probability

Statistics might be defined as **the study of uncertainty: how to measure it, and what to do about it**, and probability as the part of mathematics (and philosophy) devoted to the **quantification of uncertainty**.

The systematic study of probability is **fairly recent** in the history of ideas, dating back to about 1650 (e.g., Hacking, 1975).

In the last 350 years **three main ways to define probability** have arisen (e.g., Oakes, 1990):

- **Classical**: Enumerate **elemental outcomes** (EOs) in a way that makes them **equipossible** on, e.g., symmetry grounds, and compute  $P_C(A) = \text{ratio of } n_A = (\text{number of EOs favorable to } A) \text{ to } n = (\text{total number of EOs})$ .
- **Frequentist**: Restrict attention to attributes  $A$  of **events**: phenomena that are inherently repeatable under “identical” conditions; define  $P_F(A) = \text{limiting value of relative frequency with which } A \text{ occurs as the number of repetitions } \rightarrow \infty$ .
- **Personal**, or “**Subjective**”, or **Bayesian**: Imagine betting with someone about the truth of **proposition**  $A$ , and ask Yourself what **odds**  $O_{\text{You}}$  (in favor of  $A$ ) You would need to give or receive in order that You judge the bet fair; then (for You)  $P_{B:\text{You}}(A) = \frac{O_{\text{You}}}{(1+O_{\text{You}})}$ .

Other approaches not covered here include **logical** (e.g., Jeffreys, 1961) and **fiducial** (Fisher, 1935) probability.

# Strengths and Weaknesses

Each of these probability definitions has general **advantages** and **disadvantages**:

- **Classical**

- **Plus:** **Simple**, when applicable (e.g., idealized coin-tossing, drawing colored balls from urns, etc.).
- **Minus:** The only way to define “equipossible” without a circular appeal to probability is through the **principle of insufficient reason**—You judge EOs equipossible if You have no grounds (empirical, logical, or symmetrical) for favoring one over another—but this leads to **paradoxes** (e.g., assertion of equal uncertainty is not invariant to the choice of scale on which it’s asserted).

- **Frequentist**

- **Plus:** Mathematics relatively **tractable**.
- **Minus:** Only applies to **inherently repeatable events**, e.g.,  $P_F$ (George W. Bush will be re-elected in 2004) is (strictly speaking) **undefined**.

- **Bayesian**

- **Plus:** **All forms of uncertainty** are in principle quantifiable with this approach.
- **Minus:** There’s no guarantee that the answer You get by querying Yourself about betting odds will retrospectively be seen by You or others as **“good”** (but how should the **quality** of an uncertainty assessment itself be assessed?).

# Application to HIV Screening

$$P(A) = P(\text{this patient is HIV-positive}) = ?$$

Data are available from medical journals on **prevalence** of HIV-positivity in various subsets of  $\mathcal{P} = \{\text{all humans}\}$  (e.g., it's higher in gay people, and lower in older people).

All three probabilistic approaches require You to use Your **judgment** to identify the **recognizable subpopulation**  $\mathcal{P}_{\text{this patient}}$  (Fisher, 1956; Draper et al., 1993): this is

*the smallest subset to which this patient belongs for which the HIV prevalence differs from that in the rest of  $\mathcal{P}$  by an amount You judge as **large enough to matter in a practical sense**.*

Within  $\mathcal{P}_{\text{this patient}}$  You regard HIV prevalence as **close enough to constant** that the differences aren't worth bothering over, but the differences between HIV prevalence in  $\mathcal{P}_{\text{this patient}}$  and its complement matter to You.

Here  $\mathcal{P}_{\text{this patient}}$  might consist of everybody who matches this patient on (e.g.) **gender, age** (category, e.g., 25–29), and **sexual orientation**.

**NB** This is a **modeling choice** based on **judgment**; different reasonable people might make different choices; thus probability modeling in the real world is **inherently subjective** (see page 11 below for more details).

As a **classicist** You would then (a) use this definition to establish equipossibility within  $\mathcal{P}_{\text{this patient}}$ , (b) count  $n_A =$ (number of HIV-positive people in  $\mathcal{P}_{\text{this patient}}$ ) and  $n =$ (total number of people in  $\mathcal{P}_{\text{this patient}}$ ), and (c) compute  $P_C(A) = \frac{n_A}{n}$ .

# HIV Screening (continued)

As a **frequentist** You would (a) equate  $P(A)$  to  $P$ (a person chosen at random with replacement (i.e., independent identically distributed (**IID**) sampling) from  $\mathcal{P}_{\text{this patient}}$  is HIV-positive), (b) imagine **repeating** this random sampling indefinitely, and (c) conclude that the limiting value of the relative frequency of HIV-positivity in these repetitions would also be  $P_F(A) = \frac{n_A}{n}$ .

**NB** Strictly speaking we're not allowed in the frequentist approach to talk about  $P$ (**this patient** is HIV-positive)—either he is or he isn't.

In the frequentist paradigm we can only talk about the **process** of sampling people like him from  $\mathcal{P}_{\text{this patient}}$ .

As a **Bayesian**, with the information given here You would regard this patient as **exchangeable** (de Finetti, e.g., 1964, 1974/5) with all other patients in  $\mathcal{P}_{\text{this patient}}$ —meaning informally that You judge Yourself equally uncertain about HIV-positivity for all the patients in this set—and this judgment, together with the axioms of **coherence**, would also yield  $P_{B:\text{You}}(A) = \frac{n_A}{n}$  (although I've not yet said why this is so).

**Exchangeability** and **coherence** will be defined and explored in more detail in what follows.

Note that with the same information base the three approaches in this case have led to the same answer, although the **meaning** of that answer depends on the approach, e.g., frequentist probability describes the **process** of observing a repeatable event whereas Bayesian probability is an attempt to **quantify Your uncertainty** about something, repeatable or not.

# Subjectivity and Objectivity

The classical and frequentist approaches have sometimes been called **objective**, whereas the Bayesian approach is clearly **subjective**, and—since objectivity sounds like a good goal in science—this has sometimes been used as a claim that the classical and frequentist approaches are superior.

I'd argue, however, that in interesting applied problems of realistic complexity, the **judgment of equivalence or similarity** (equipossibility, IID, exchangeability) that's central to all three theories makes them all subjective in practice.

Imagine, for example, that You were given data on HIV prevalence in a large group of people, along with many variables (possible **predictors**) that might or might not be relevant to identifying the recognizable subpopulations.

You and other reasonable people working independently might well differ in your judgments on **which of these predictors are relevant** (and how they should be used in making the prediction), and the result could easily be **noticeable variation** in the estimates of  $P(\text{HIV positive})$  obtained by You and the other analysts, even if you and the other people all attempt to use “objective” methods to arrive at these judgments (there are many such methods, and they don't always lead to the same conclusions).

Thus the assessment of complicated probabilities is **inherently subjective**—there are “**judgment calls**” built into probabilistic and statistical analysis.

With this in mind attention in all three approaches should evidently shift away from trying to achieve “objectivity” toward two things: (1) the explicit statement of the **assumptions** and **judgments** made on the way to Your probability assessments, so that other people may consider their plausibility, and (2) **sensitivity analyses** exploring the mapping from assumptions to conclusions.

(To a Bayesian saying that  $P_B(A)$  is **objective** just means that lots of people more or less agree on its value.)

## 1.2 Sequential Learning; Bayes' Theorem

Let's say that, with this patient's values of relevant demographic variables, the prevalence of HIV estimated from the **medical literature**,  $P(A) = P(\text{he's HIV-positive})$ , in his recognizable subpopulation is about  $\frac{1}{100} = 0.01$ .

To improve this estimate by **gathering data specific to this patient**, You decide to take some blood and get a result from *ELISA*.

Suppose the test comes back positive—what is Your **updated**  $P(A)$ ?

Bayesian probability has that name because of the simple **updating rule** attributed to Thomas Bayes (1763), who was one of the first people to define conditional probability and make calculations with it:

**Bayes' Theorem  
for propositions**

$$P(A|D) = \frac{P(A) P(D|A)}{P(D)}$$

(actually—Stigler, 1986; Bernardo and Smith, 1994—Bayes only stated and worked with a **special case** of this; the general form was first used by Laplace, 1774).

In the usual application of this  $A$  is an **unknown quantity** (such as the truth value of some proposition) and  $D$  stands for some **data** relevant to Your uncertainty about  $A$ :

$$P(\text{unknown}|\text{data}) = \frac{P(\text{unknown}) P(\text{data}|\text{unknown})}{\text{normalizing constant}}$$

posterior =  $c \cdot$  prior  $\cdot$  likelihood

# Bayes' Theorem (continued)

The terms **prior** and **posterior** emphasize the sequential nature of the learning process:  $P(\text{unknown})$  was Your uncertainty assessment before the data arrived; this is updated multiplicatively on the probability scale by the **likelihood**  $P(\text{data}|\text{unknown})$ , and renormalized so that total probability remains 1.

Writing the Theorem both for  $A$  and (not  $A$ ) and combining gives a (perhaps even **more**) **useful** version:

## Bayes' Theorem in odds form

$$\frac{P(A|\text{data})}{P(\text{not } A|\text{data})} = \frac{P(A)}{P(\text{not } A)} \cdot \frac{P(\text{data}|A)}{P(\text{data}|\text{not } A)}$$

$$\text{posterior odds} = \text{prior odds} \cdot \text{Bayes factor}$$

Other names for the Bayes factor are the **data odds** and the **likelihood ratio**, since this factor measures the relative plausibility of the data given  $A$  and (not  $A$ ).

Applying this to the HIV example requires additional information about *ELISA* obtained by screening the blood of people with **known HIV status**:

$$\begin{aligned} \text{sensitivity} &= P(\text{ELISA positive}|\text{HIV positive}) \quad \text{and} \\ \text{specificity} &= P(\text{ELISA negative}|\text{HIV negative}) \end{aligned}$$

In practice *ELISA*'s operating characteristics are (or at least seem) **rather good**—sensitivity about 0.95, specificity about 0.98—so you might well expect that

$$P(\text{this patient HIV positive}|\text{ELISA positive})$$

would be **close to 1**.

# Inference and Decision-Making

Here the updating produces a **surprising result** (if you've never seen this sort of thing before): the Bayes factor comes out

$$B = \frac{\text{sensitivity}}{1 - \text{specificity}} = \frac{0.95}{0.02} = 47.5,$$

which sounds like **strong evidence** that this patient is HIV positive, but the prior odds are quite a bit stronger the other way ( $\frac{P(A)}{1-P(A)} = 99$  to 1 **against** HIV), leading to posterior

$$\text{odds of } \frac{99}{47.5} \doteq 2.08 \text{ **against** HIV, i.e.,}$$
$$P(\text{HIV positive}|\text{data}) = \frac{1}{1+\text{odds}} = \frac{95}{293} \doteq 0.32 (!).$$

The reason for this is that *ELISA* was designed to have a vastly better **false negative** rate— $P(\text{HIV positive}|\text{ELISA negative}) = \frac{5}{9707} \doteq 0.00052 \doteq 1$  in 1941—than **false positive** rate— $P(\text{HIV negative}|\text{ELISA positive}) = \frac{198}{293} \doteq 0.68 \doteq 2$  in 3.

This in turn is because *ELISA*'s developers judged that it's **far worse to tell somebody who's HIV positive that they're not than the other way around** (reasonable for using *ELISA* for, e.g., **blood bank screening**).

This false positive rate would make widespread screening for HIV based only on *ELISA* a **truly bad idea**.

Formalizing the consequences of the two types of error in diagnostic screening would require quantifying **misclassification costs**, which shifts the focus from (scientific) **inference** (the acquisition of knowledge for its own sake: Is this patient really HIV-positive?) to **decision-making** (putting that knowledge to work to answer a public policy or business question, e.g.: What use of *ELISA* and *Western Blot* would yield the **optimal screening strategy?**).

## 1.3 Bayesian Decision Theory

Axiomatic approaches to **rational decision-making** date back to Ramsay (1926), with von Neumann and Morgenstern (1944) and Savage (1954) also making major contributions.

The ingredients of a **general decision problem** (e.g., Bernardo and Smith, 1994) include

- A set  $\{a_i, i \in I\}$  of available **actions**, one of which You will choose;
- For each action  $a_i$ , a set  $\{E_j, j \in J\}$  of **uncertain outcomes** describing what will happen if You choose action  $a_i$ ;
- A set  $\{c_j, j \in J\}$  of **consequences** corresponding to the outcomes  $\{E_j, j \in J\}$ ; and
- A **preference relation**  $\leq$ , expressing Your preferences between pairs of available actions ( $a_1 \leq a_2$  means “ $a_1$  is not preferred by You to  $a_2$ ”).

Define  $a_1 \sim a_2$  (“ $a_1$  and  $a_2$  are **equivalent**” to You) iff  $a_1 \leq a_2$  and  $a_2 \leq a_1$ .

This preference relation induces a **qualitative ordering** of the uncertain outcomes ( $E \leq F$  means “ $E$  is not more likely than  $F$ ”), because if You compare two dichotomized possible actions, involving the same consequences and differing only in their uncertain outcomes, the fact that You prefer one action to another means that You must judge it more likely that if You take that action **the preferred consequence will result.**

# Coherence

Within this framework You have to make further assumptions—the **coherence** axioms—to ensure that Your actions are internally consistent.

**Informally** (see Bernardo and Smith, 1994, for the formalism) these are:

- An axiom insisting that You be willing to express preferences between simple **dichotomized** possible actions ( $\{a, \text{not } a\}$ );
- A **transitivity** axiom in which (for all actions  $a, a_1, a_2, a_3$ )  $a \leq a$ , and if  $a_1 \leq a_2$  and  $a_2 \leq a_3$  then  $a_1 \leq a_3$ ; and
- An axiom based on the **sure-thing principle**: if, in two situations, no matter how the first comes out the corresponding outcome in the second is preferable, then You should prefer the second situation overall.

This puts  $\leq$  on a sound footing for **qualitative** uncertainty assessment, but does not yet imply how to quantify—it's like being able to say that one thing weighs less than another but not to say by how much.

To go further requires a fourth assumption, analogous to the existence of a set of **reference standards** (e.g., an official kg weight, half-kg, etc.) and the ability to make arbitrarily precise comparisons with these standards:

- An axiom guaranteeing that for each outcome  $E$  there exists a **standard outcome**  $S$  (e.g., “idealized coin lands heads”) such that  $E \sim S$ .

This framework implies the existence and uniqueness of a (personal) probability  $P_{B:Y_{\text{OU}}}$  (abbreviated  $P$ ), mapping from outcomes  $E$  to  $[0,1]$  and corresponding to the judgments in

Your definition of  $\leq$ , and a **utility function**  $U_{Y_{\text{OU}}}$  (abbreviated  $U$ ; large values preferred, say), mapping from consequences  $c$  to  $R$  and quantifying Your preferences.

# “Dutch Book”

This has all been rather **abstract**.

**Three concrete results** arising from this framework may make its implications clearer:

- Bayes’ original definition of personal probability is helpful in thinking about how to quantify uncertainty. Pretending that consequences are monetary (e.g., US\$), to say that  $P_{B:You}(E) = p$  for some uncertain outcome  $E$  whose truth value will be known in the future is to say that You’re **indifferent** between (a) receiving  $\$p \cdot m$  for sure (for some hypothetical (and reasonably small) amount of money  $\$m$ ) and (b) betting with someone in such a way that you will get  $\$m$  if  $E$  turns out to be true and nothing if not (you can use this to estimate  $P_{B:You}(E)$ ).
- Any coherent set of probability judgments *must satisfy the standard axioms and theorems of a finitely additive probability measure*:
  - $0 \leq P(E) \leq 1$  and  $P(E^c) = 1 - P(E)$ ;
  - $P(E_1 \text{ or } \dots \text{ or } E_J) = \sum_{j \in J} P(E_j)$  for any finite collection  $\{E_j, j \in J\}$  of disjoint outcomes;
  - $P(E \text{ and } F) = P(E) \cdot P(F)$  for any two independent outcomes (informally,  $E$  and  $F$  are **independent** if Your uncertainty judgments involving one of them are unaffected by information about the other); and
  - **Conditional probability** has a natural definition in this setup, corresponding to the updating of Your uncertainty about  $E$  in light of  $F$ , and with this definition 
$$P(E|F) = \frac{P(E \text{ and } F)}{P(F)}.$$

Otherwise (de Finetti, 1964) someone betting with You on the basis of Your probability judgments can **make “Dutch book” against you**, i.e., get You to agree to a series of bets that are guaranteed to lose You money.

Thus coherent Bayesian probability **obeys the same laws** as with the classical and frequentist approaches (apart from a technical issue about finite versus countable additivity).

# Maximization of Expected Utility

- Nothing so far has said clearly what choice to make in a decision problem if You wish to **avoid incoherence**.

If the outcomes were certain You'd evidently choose the action that **maximizes Your utility function**, but since they're not the best action must involve weighing both Your probabilities for the uncertain outcomes and the utilities You place on their consequences.

It's a direct implication of the framework here that the form this weighing should take is **simple** and **clear**:

## Maximization of Expected Utility (MEU)

Given Your probability and utility judgments, Your decision-making is coherent iff for each action  $a_i$ , with associated uncertain outcomes  $\{E_j, j \in J\}$  and consequences  $\{c_j, j \in J\}$ , You compute the **expected utility**  $EU_i = \sum_{j \in J} U(c_j)P(E_j)$  and choose the action that **maximizes**  $\{EU_i, i \in I\}$ .

**Example: HIV screening.** As a simplified version of this problem consider choosing between two actions:

- $a_1$ : Obtain *ELISA* results at a cost of  $c_1 = \$20$ ; if **positive** conclude this patient is HIV+, if **negative** conclude HIV-.
- $a_2$ : Same as  $a_1$  except if *ELISA* comes out **positive**, obtain *Western Blot (WB)* results at an additional cost of  $c_2 = \$100$ ; if *WB* is **positive** conclude HIV+, if **negative** conclude HIV-.

# HIV Screening

With action  $a_1$  the **probabilities, uncertain outcomes, and utilities** are as follows:

Probability	True HIV Status	ELISA Status	Utility
.0095	+	+	$-c_1$
.0005	+	-	$-c_1 - L_I$
.0198	-	+	$-c_1 - L_{II}$
.9702	-	-	$-c_1$

Here  $L_I$  and  $L_{II}$  are the **false negative (false positive)** monetary losses suffered by this patient if he really is HIV+ (HIV-) but *ELISA* says he is HIV- (HIV+).

The **expected utility** with action  $a_1$  is thus

$$\begin{aligned}
 EU_1 &= .0095(-c_1) + .0005(-c_1 - L_I) + \dots + .9702(-c_1) \\
 &= -(c_1 + .0005L_I + .0198L_{II}) .
 \end{aligned}$$

The **corresponding table** for action  $a_2$  is:

Prob.	True HIV Status	ELISA Status	WB Status	Utility
.00945	+	+	+	$-c_1 - c_2$
.00005	+	+	-	$-c_1 - c_2 - L_I$
.00004	+	-	+	$-c_1 - L_I$
.00046	+	-	-	$-c_1 - L_I$
.0001	-	+	+	$-c_1 - c_2 - L_{II}$
.0197	-	+	-	$-c_1 - c_2$
.00095	-	-	+	$-c_1$
.96925	-	-	-	$-c_1$

## HIV Screening (continued)

These probabilities arise from *WB*'s design (the goal was to have about the same false negative rate as *ELISA* and a much lower false positive rate (about 0.1), leading to a **slightly worse sensitivity** (0.949) but **much improved specificity** (0.999)).

The **expected utility** with action  $a_2$  comes out

$$\begin{aligned} EU_2 &= .00945(-c_1 - c_2) + \dots + .9604(-c_1) \\ &= -(c_1 + .0293c_2 + .00055L_I + .0001L_{II}) . \end{aligned}$$

By **MEU** You should prefer  $a_2$  to  $a_1$  iff  $EU_2 > EU_1$ , i.e., iff

$$\boxed{.0197L_{II} - .00005L_I - .0293c_2 > 0 .}$$

Thus  $a_2$  becomes more desirable as the loss suffered with a false positive (negative) increases (decreases), and less desirable as *WB*'s cost increases, all of which **makes good sense**.

It's interesting to note that with a modest value for  $L_{II}$  (e.g., \$1,000), the monetary advantage from taking action  $a_2$  is **quite small** even with a realistically huge value for  $L_I$  (e.g., \$100,000, which leads to an edge for  $a_2$  of only about \$12).

This is due to the **extremely low false negative rate** for both tests— $L_I$  would have to be over \$335,000 for  $a_1$  to dominate!

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