

# Human-Assisted Graph Search: It’s Okay to Ask Questions

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## ABSTRACT

We consider the problem of *human-assisted graph search*: given a directed acyclic graph with some (unknown) target node(s), we consider the problem of finding the target node(s) by asking an omniscient human questions of the form “Is there a target node that is reachable from the current node?”. This general problem has applications in many domains that can utilize human intelligence, including curation of hierarchies, debugging workflows, image segmentation and categorization, interactive search and filter synthesis. To our knowledge, this work provides the first formal algorithmic study of the optimization of human computation for this problem. We study various dimensions of the problem space, providing algorithms and complexity results. We also compare the performance of our algorithm against other algorithms, for the problem of webpage categorization on a real taxonomy. Our framework and algorithms can be used in the design of an optimizer for crowd-sourcing platforms such as Mechanical Turk.

## 1. INTRODUCTION

Crowd-sourcing services, such as Amazon’s Mechanical Turk [2] and CrowdFlower [1] allow organizations to set up tasks that humans can perform for a certain reward. The goal is to harness “human computation” in order to solve problems that are very difficult to tackle completely algorithmically. Examples of such problems in practice include object recognition, language understanding, text summarization, ranking, and labeling [9].

In a typical crowd-sourcing setting, the tasks are broken down to simple questions (often with a YES/NO answer) that can be easily tackled by humans. Since each question comes at a price, be it money, effort, or time, it is desirable to minimize the number of questions that need to be answered in order to achieve the overall objective. Thus, we would like a general-purpose *human computation optimizer* that selects the specific questions to be asked so as to minimize some cost metric. We develop core algorithms for such an optimizer, considering a class of *human-assisted graph search* problems.

In human-assisted graph search, or HumanGS for short, we are given as input a directed acyclic graph that contains a set of un-

known *target* nodes (collectively called the *target set*), and the goal is to discover the identities of these target nodes solely by asking *search questions* to humans. A search question is a question of the form “Given a node  $x$  in the graph, is there a target node reachable from node  $x$  via a directed path?”. (Note that, by definition, each search question corresponds to a node in the graph, and we can ask a search question at any node.) The objective is to select the optimal set of nodes at which to ask search questions in order to ascertain the identities of the target nodes. Later, we examine different notions of optimality, including minimizing the total number of questions, or minimizing the set of resulting possible target nodes after a fixed number of questions.

**EXAMPLE 1.1.** *Suppose our goal is to categorize an image into one of the classes of the hierarchical taxonomy shown in Fig. 1(a). If the image is that of a Nissan car, but the model is not identifiable, the most suitable category is “Nissan”. If the model is identifiable as well, say, Sentra, then the most suitable category would be “Sentra”. Since categorization of images is a task that can be performed better by humans than by computers, we would like to utilize human intelligence. We wish to ascertain the most suitable category (which could be anywhere in the taxonomy) by asking the minimum number of questions to humans.*

*This task is an instance of the HumanGS problem. The taxonomy is the DAG, with each category corresponding to a node. The target node is the most suitable category of the image. Asking a search question at a node/category  $X$  is equivalent to asking a question of the form “Is this an  $X$ ?”. For instance, in the Nissan car example above, receiving a YES answer to “Is this a **Car**?” says that the most suitable category, in this case **Nissan**, is reachable from the category **Car**. Also, asking a question at the root, **Vehicle**, (a general question) gives a YES answer while asking a question at a leaf, **Maxima** (a specific question), gives a NO answer. If the image is that of a car, but the model is not identifiable, then asking a question at **Vehicle** and **Car** will yield YES, while all other questions will receive a NO answer. □*

There are several interesting properties that make HumanGS a nontrivial problem. First, the answers to different search questions may be correlated, e.g., if the answer to the search question at a node is YES, then the answer to a search question at an ancestor of that node will be YES as well. Therefore, it is possible to identify the target nodes without asking search questions at all nodes in the graph. Second, the location of a node affects the amount of information that can be obtained from the corresponding search question. Asking search questions at nodes close to leaves (very *specific* questions) are more likely to receive a negative answer, while asking questions at nodes close to roots (very *general* questions) are more likely to receive positive answers. Asking search questions at the “middle” nodes may give more information. In this sense,

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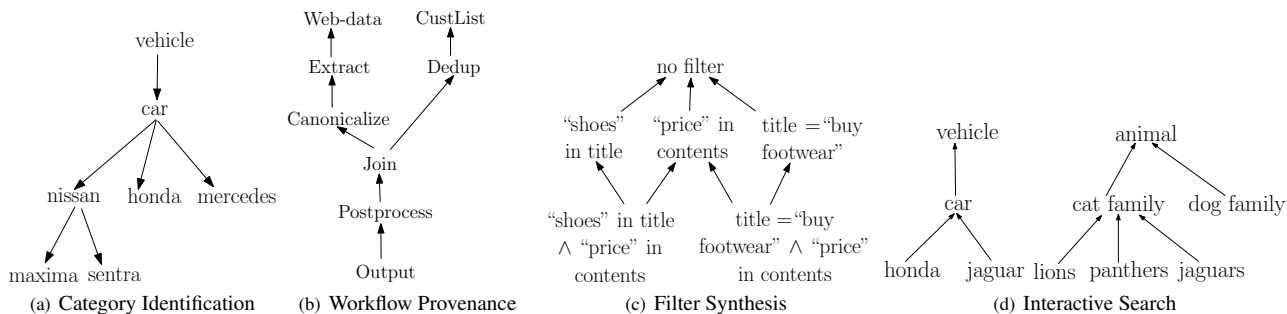


Figure 1: Examples of HumanGS

the HumanGS problem is similar to the 20 questions game<sup>1</sup>, where very specific or very general questions do not help.

An additional challenge stems from the usage model of crowd-sourcing services in practice. Ideally, we would like to issue one question at a time, selecting the next search question based on the answers to previous questions. However, typical crowd-sourcing services incur a high latency for obtaining the answer to a single question: the question first has to be posted to the service, then a human worker has to find the question and decide to answer it, and finally the answer is sent back to the requester. This characteristic leads one away from the sequential one-question-at-a-time model. A more common model is to issue several questions in parallel, whose answers are then combined to solve the task at hand. The challenge, therefore, is to reason about the possible answers for search questions at different nodes in the graph, and to select the set of questions in order to infer as much information as possible about the target nodes *across all possibilities*. In this paper we study this “offline” model analytically. However, in Section 6 (Experiments) we briefly look at a hybrid approach, where we ask some questions, examine the results, and then ask additional questions, but a full theoretical study of a hybrid model is left for future work.

In this paper, we develop algorithms that compute the optimal set of questions for different variants of the HumanGS problem. Our primary contributions can be summarized as follows:

- We identify a number of real-world instances of the HumanGS problem, which lead to the delineation of three orthogonal problem dimensions. (Section 2)
- We formally define the HumanGS problem. (Section 3)
- We present algorithms and complexity results for the problem for the three dimensions identified in Section 2. We show that while the general problem is computationally hard, the more constrained variants are tractable. (Sections 4 and 5)
- We study the performance of our algorithms versus other algorithms for an instance of HumanGS on real-world data. (Section 6)

To the best of our knowledge, ours is one of the first papers to address the problem of optimizing a computation that harnesses human “processors” through a crowd-sourcing service. Requesting input from humans as a component of a computer algorithm is not new; the field of active learning [23] also considers the problem of optimally requesting input from experts. However, this input is typically for generating test datasets for machine learning problems.

Other studies examine different aspects of human computation technologies, such as social issues and application usage. HumanGS bears some resemblance to classification tasks solved by decision trees, but in our case no training set, statistics, or attributes are provided, and the only questions that can be asked involve reachability. We discuss related work in more detail in Section 7.

<sup>1</sup>20 questions is a two player game, where one player thinks of an object, person or place and the other player has to guess the identity of that item by asking the first player up to 20 YES/NO questions.

## 2. APPLICATIONS AND DIMENSIONS

We will study variants of the HumanGS problem along three orthogonal dimensions. The choice of these dimensions is driven by applications that represent instantiations of HumanGS in practice. We discuss the applications first and then formally define the derived dimensions.

- **Image Categorization:** Described in Example 1.1.
- **Manual Curation:** (Isomorphic to image categorization.) In an existing taxonomy (such as Wikipedia, phylogenetic trees, web of concepts [15]), we wish to manually insert new concepts, topics and items to their most suitable location in the hierarchy. Manual curation of each new item can be reduced to an instance of HumanGS in a manner similar to image categorization. A search question at  $x$  corresponds to the question “Is the item a kind of  $x$ ?”, while the target node is the most suitable parent of the new item.
- **Debugging of Workflows:** Suppose that we detect an erroneous result in the output of a workflow. Naturally, we would like to detect the earliest workflow steps that introduced the error. Assume that we maintain provenance information for the workflow [19, 24], so that we can identify the fragment of the output of each workflow step that is linked to the erroneous result. If we view the workflow as a DAG (with the direction of the edges reversed), as in Fig. 1(b), then isolating the earliest incorrect workflow steps can be reduced to an instance of HumanGS. A search question at  $x$  corresponds to asking the user “Is the output fragment at point  $x$  wrong?”, and the target nodes are the earliest steps in the workflow that introduced errors into the resulting output.
- **Filter Synthesis:** Suppose that a user wishes to apply a filter on a data set as part of some ad-hoc analysis. We can help the user formulate the filter by asking them a few questions. If we arrange the candidate filters in a DAG as in Fig. 1(c), (Candidate filters can be extracted by analyzing the data set [3]) then filter synthesis can be reduced to an instance of HumanGS. Edges in the DAG indicate logical implication between filter fragments. A search question at  $x$  corresponds to asking the user “Do you want all data items satisfying condition  $x$  to be part of the result?”. The target set comprises all filters whose disjunction yields the intended filter.
- **Interactive Search:** In interactive search [8, 17], the search engine asks the user a few questions that help isolate the concepts that best encompass his/her information need. The questions are based on a backend hierarchy of concepts that cover the crawled web-data. Fig. 1(d) shows an example hierarchy. (Note that the edges in the hierarchy go from more specific to more general concepts.) After presenting the user with the initial results, the search engine can pose questions of the form “Do you want more results like concept  $X$ ?”. The target nodes are the most general concepts that the user is interested in (i.e., those that encompass his information need).

Inspired by these applications, we derive three dimensions that characterize the different instances of the HumanGS problem.

### Dimension 1: Single/Multi.

The first dimension controls the characteristics of the target set.

In the Single variant, the target set contains a single node. The Multi variant does not constrain the size of the target set.

While Single is relevant for Image Categorization and Manual Curation, the Multi variant is relevant for the other three applications listed above. As a concrete example, for a given incorrect result at the output in Figure 1(b), both the canonicalization step on web-data as well as the deduping of the customer list could be introducing errors. As another example, consider creating a filter for web-pages relating to shoe shopping, as in Figure 1(c). Here, a user might be interested in pages that contain “shoes” in the title and “price” in the contents, or pages that have as a title “buy footwear”.

### Dimension 2: Bounded/Unlimited.

The second dimension controls the number of questions that can be asked. In the Bounded case, we are given a budget  $k$  for the total number of questions that can be asked and we want to compute a node set  $N$ ,  $|N| \leq k$ , at which to ask search questions such that we narrow down the candidates for the target set as much as possible. The Unlimited case does not put a bound on the number of questions. In this case, we want to compute the minimal set of nodes to ask questions such that we precisely identify the target set.

The Bounded case is relevant when asking questions is potentially costly, e.g., on Mechanical Turk, and we wish to bound the total cost while narrowing down the possibilities for the target nodes. The Bounded case is also relevant in Interactive Search, where it is not practical to ask an unlimited number of questions (since the user may not be willing to answer too many questions). Once we receive the answers to the questions, we can display results related to the concepts that may correspond to target nodes. The results can still be useful for the user, even if we do not identify the target set precisely. The Unlimited case is relevant whenever an exact answer is required, such as Manual Curation of hierarchies.

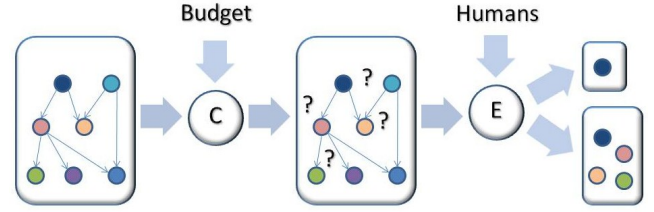
### Dimension 3: DAG/Downward-Forest/Upward-Forest.

The third dimension controls the type of DAG on which we perform the search. Besides a general DAG, which is relevant for Filter Synthesis (as in Fig. 1(c)) and Debugging of Workflows, we consider two restricted structures. The first is a “downward forest” structure, where there are several trees with edges directed from parents to children, as in Fig. 1(a). This structure is relevant in Image Categorization and Manual Curation. The second structure is an “upward forest”, which is the same as a downward-forest except that the edges are reversed, as in Fig. 1(d). This structure is relevant in Interactive Search.

## 3. THE HumanGS PROBLEM

Informally, our approach can be summarized using Fig. 2 (which illustrates HumanGS for the Single case of Dimension 1.) We first select a set of questions to ask via HumanGS procedure  $C$ . Subsequently, via an evaluation procedure  $E$ , we obtain answers from humans to the selected questions, using which we compute the possible candidates for the target node, which can be either the target node or a superset. We now attempt to formalize these intuitions.

We are given a directed acyclic graph  $G = (V, E)$  that reflects the semantics of a specific instance of HumanGS. We use  $n \equiv |V|$  to denote the number of nodes in the graph. A node  $v \in V$  is *reachable* from another node  $u \in V$  if there exists a directed path from  $u$  to  $v$ . The *reachable set* of  $u$ , denoted  $\text{rset}(u)$ , contains all nodes that are reachable from  $u$ , including  $u$ . For instance, the reachable set of nissan in Figure 1(a) is {nissan, maxima, sentra}. The *preceding set* of  $u$ , denoted  $\text{pset}(u)$ , contains all  $v \neq u$  such that  $u \in \text{rset}(v)$ . For instance, the preceding set of nissan in Figure 1(a) is {vehicle, car}. We say that  $u$  and  $v$  are *unrelated* if there



**Figure 2: The HumanGS procedure  $C$  selects a set of nodes to ask questions. Once Humans answer the questions via the evaluator procedure  $E$ , we may be able to isolate the target node, or a set of nodes one of which is the target node, depending on the answers.**

is no directed path between them, i.e.,  $u \notin \text{pset}(v) \cup \text{rset}(v)$ .

We assume that the HumanGS instance involves a node-set  $U^* \subseteq V$ , termed the *target set*, that comprises the *target nodes*. The target set must satisfy the following property:

*Independence Property:* No two nodes in  $U^*$  are related.

This property holds in all motivating applications in Section 2. Intuitively, if there are two nodes  $u$  and  $v \in U^*$  such that  $u \neq v$  and  $u \in \text{rset}(v)$ , then  $v$  can be discarded because  $u$  “subsumes”  $v$ . For instance, in Figure 1(d), if the user is interested in ‘cat family’ as well as ‘lions’, then we would prefer to retain ‘cat family’ in  $U^*$  instead of ‘lions’ because ‘cat family’ subsumes ‘lions’. (Note that the independence property holds even in the workflow debugging application: if there are multiple related error-causing steps, we identify the earliest ones.)

We can informally describe the HumanGS problem as computing a set of nodes  $\{u_1, \dots, u_k\}$  such that the answers to the corresponding *search questions* at the set of nodes lead to the identification of  $U^*$ . Note that each search question corresponds to a node in the graph, and hence we interchangeably use “asking a question”, “asking a question at a node” and “asking a node”. Asking a question is defined formally as follows.

**DEFINITION 3.1 (ASKING A QUESTION  $q(u, U^*)$ ).** *Asking a question at node  $u \in V$ , denoted as  $q(u, U^*)$ , returns YES if  $\text{rset}(u) \cap U^* \neq \emptyset$ , and NO otherwise.*

In other words,  $q(u, U^*)$  returns YES iff a directed path starting at  $u$  reaches at least one node in  $U^*$ . Note that  $u$  may itself be in the target set. Also note that for any  $U_1^*$  and  $U_2^*$ ,  $U_1^* \neq U_2^*$ , both of which satisfy the independence property, there is some node at which asking a question would give different answers. (Consider  $u$  such that  $u \in U_1^*$  and  $u \notin U_2^*$ . Either there is no  $v \in \text{rset}(u)$  present in  $U_2^*$ , in which case asking a question at  $u$  would give different answers. Or, there is such a  $v \neq u$ , in which case asking a question at  $v$  would give different answers.)

A solution to HumanGS is always feasible, as we can identify  $U^*$  by asking questions at every node in  $V$ . However, it is not necessary to ask questions at every node, as the following trivial lemma illustrates.

**LEMMA 3.2 (DAG PROPERTY).** *If  $q(u, U^*)$  is YES, then  $q(v, U^*)$  is YES for every  $v$  in  $\text{pset}(u)$ . Conversely, if  $q(u, U^*)$  is NO then  $q(v, U^*)$  is NO for every node  $v$  in  $\text{rset}(u)$ .*

Let  $N \subseteq V$  be some set of nodes at which we ask questions. In general, the answers to these questions may not be sufficient to precisely identify  $U^*$ , since there may be other nodes (not in  $U^*$ ) for which the current answers would be the same even if those nodes were in  $U^*$ . We introduce the notion of a *candidate set* to capture the possibilities for  $U^*$  based on questions on a node-set  $N$ . The candidate set, denoted as  $\text{cand}(N, U^*)$ , is the maximal set of nodes that we cannot distinguish from  $U^*$  based solely on the answers to questions at the nodes in  $N$ . In other words, the nodes in  $\text{cand}(N, U^*)$  have the same reachability properties (with respect

		DAG	Downward-Forest	Upward-Forest	Downward Bal.	Upward Bal.
Single	Bounded	NP-Complete( $n, k$ ), $O((2n)^k n^2 k)$	$O(n \log n)$	$O(mk^2 n^5)$	$O(1)$	$O(1)$
	Unlimited	$\min\{\log^2 o, \log n\} \times$ Single-Bounded	$O(1)$	$O(n)$	$O(1)$	$O(n)$
Multi	Bounded	NP-Hard( $n, k$ ), $\Sigma_2^P(n, k)$	$O(mk^2 n^6)$		$O(1)$	
	Unlimited		$O(1)$			

**Table 1: Summary of Results. (Bal. stands for Balanced Trees):  $k$  is the budget of questions,  $n = |V|$ ,  $m$  is the arity of the tree or forest, and  $o$  is the size of the optimal  $N$ .**

to  $N$ ) as the nodes in  $U^*$ . For  $|N| = 0$ , we have the trivial result that  $\text{cand}(N, U^*) = V$ . We first consider how asking a single question (i.e.,  $|N| = 1$ ) allows us to restrict the contents of the candidate set beyond  $V$ .

**THEOREM 3.3 (ONE QUESTION PRUNING).** *Assume that we ask a single question at node  $u$ . The candidate set is computed as follows, based on the answer and the variant of Single/Multi that the HumanGS instance falls under:*

$$\text{cand}(\{u\}, U^*) = \begin{cases} V - \text{rset}(u) & \text{q}(\{u\}, U^*) = \text{NO} \\ V - \text{pset}(u) & \text{q}(\{u\}, U^*) = \text{YES} \wedge \text{Multi} \\ \text{rset}(u) & \text{q}(\{u\}, U^*) = \text{YES} \wedge \text{Single} \end{cases}$$

**PROOF.** If we get a NO on asking a question at a node  $u$ , then none of  $\text{rset}(u)$  can be present in  $\text{cand}(\{u\}, U^*)$ , but any other node could be a target node, therefore  $\text{cand}(\{u\}, U^*) = V - \text{rset}(u)$ . Suppose we get YES, so a target node exists in  $\text{rset}(u)$ . If there are one or more target nodes ( $|U^*| \geq 1$ , as in Multi), the independence property implies that no target node can be present in  $\text{pset}(u)$ . Hence,  $\text{cand}(\{u\}, U^*) = V - \text{pset}(u)$ . Furthermore, if we know that there exists a single target node, i.e.,  $|U^*| = 1$ , as in Single, then the candidate set is simply  $\text{rset}(u)$ .  $\square$

Given this base case of one question, we can compute  $\text{cand}(N, U^*)$  for a general node-set  $N$  as the intersection of the candidate sets resulting from individual questions.

**THEOREM 3.4.** *After asking questions at all nodes in a node-set  $N$ , we have:  $\text{cand}(N, U^*) = \bigcap_{u \in N} \text{cand}(\{u\}, U^*)$ .*

**PROOF.** Let  $C = \bigcap_{u \in N} \text{cand}(\{u\}, U^*)$ . If  $x \notin \text{cand}(\{u\}, U^*)$  for nodes  $x \in V$  and  $u \in N$ , then it is easy to see that  $x$  cannot be present in  $\text{cand}(N, U^*)$ . Now consider a node  $x \in C$ . For the Multi variant, consider a set  $U$  that contains  $x$ , no ancestors or descendants of  $x$ , as well as all other nodes in  $C$  that do not have descendants in  $C$ . Note that  $U$  satisfies the independence property. If  $U$  were the target set, it would give rise to the same answers to questions at  $N$  as  $U^*$ . Thus,  $x$  could be in  $U^*$  and has to be present in  $\text{cand}(N, U^*)$ . For the Single variant, consider a set  $U = \{x\}$ .  $U$  would give rise to the same answers to questions at  $N$  as  $U^*$ . Thus,  $x$  could be in  $U^*$  and has to be present in  $\text{cand}(N, U^*)$ .  $\square$

Thus, each question may enable some additional pruning of the candidate set, and the order in which questions are asked does not affect the final result. As an example, let us consider again the HumanGS problem illustrated in Fig. 1(a). Suppose that the single target node is *maxima* (recall that  $|U^*| = 1$  for this search task), and assume that we ask questions at  $N = \{\text{car}, \text{nissan}, \text{mercedes}\}$ . Clearly, the questions at *car* and *nissan* yield YES, whereas the question at *mercedes* yields NO. Based on these answers, we can assert that  $\text{cand}(N, U^*) = \{\text{nissan}, \text{maxima}, \text{sentra}\}$ . The candidate set contains the target node as well as two “false positives” (*nissan* and *sentra*). Picking  $N$  so as to minimize the number of false positives is the goal of the algorithms that we present later.

Since we are operating in an “offline” setting where the answers to previous questions are not provided to us, we are interested in minimizing the size of the candidate set in the worst case. Given that

$U^*$  is unknown, we may use the maximum size of  $\text{cand}(N, U^*)$  (under all admissible possibilities for  $U^*$ ) as an indication of the worst-case uncertainty that remains after asking the questions in  $N$ . We use  $\text{wcase}(N)$  to denote this worst-case size. A natural objective is to select  $N$  so that  $\text{wcase}(N)$  is minimized. We define  $\text{wcase}$  formally in Section 4 for Single and in Section 5 for Multi.

### 3.1 Summary of Results and Outline

Table 1 summarizes our results on the complexity of the examined variants. The details of the analysis and the corresponding algorithms are given in the following sections. The presentation is organized in two sections based on Dimension 1: Single is covered in Section 4, and Multi is covered in Section 5. Each row in the table corresponds to a subsection in the corresponding section. In each case, we first provide a formal definition of the corresponding HumanGS problem, followed by the complexity analysis for the different graph structures. In addition to the structures listed in Dimension 3, we also consider the special case of balanced trees in Appendices A.8, A.11, C.5 and C.6, which admit very efficient solutions.

We omit some proofs from the main body of the text. These proofs can be found in the Appendix.

## 4. SINGLE TARGET NODE

In the Single problem, we have the constraint that there is a single target node, i.e.,  $|U^*| = 1$ . Let this node be  $u^*$ . To simplify notation for Single, we use  $\text{cand}(N, u^*)$ , instead of  $\text{cand}(N, \{u^*\})$ , to denote the candidate set after questions have been asked at the node set  $N$ , and we use  $\text{q}(u, u^*)$ , instead of  $\text{q}(u, \{u^*\})$ , to denote the answer to asking a question at  $u$ . Recall that as in Theorem 3.3, asking a question at  $u$  for the Single problem tells us whether the candidate set  $\text{cand}(\{u\}, u^*)$  is  $\text{rset}(u)$  (if the answer is YES) or  $V - \text{rset}(u)$  (if the answer is NO).

Given a node set  $N$  at which we ask questions, we define the *worst-case candidate set size* as

$$\text{wcase}(N) = \max_{u_i \in V} |\text{cand}(N, u_i)| \quad (1)$$

In other words,  $\text{wcase}$  computes the size of the largest candidate set when the target node could be any node in  $V$ .

### 4.1 Single-Bounded

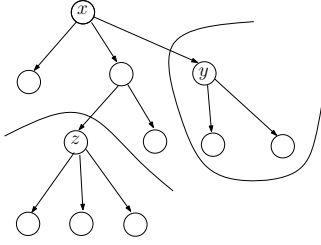
In the Single-Bounded variant we have a fixed budget  $k$  on the number of questions that may be asked, i.e., the size of  $N$  cannot exceed  $k$ . The goal is to pick the set of nodes  $N$  such that the worst-case candidate set size is minimized.

**DEFINITION 4.1 (SINGLE-BOUNDED).** *(Bounded Search for a Single target node.) Given a parameter  $k$  and the restriction that  $|U^*| = 1$ , find a set  $N$  of nodes  $N \subseteq V$  to ask questions such that  $|N| = k$  and  $\text{wcase}(N)$  is minimized.*

The following subsections examine the complexity of the problem under the different possibilities for the structure of  $G$ , i.e., general DAG, downward-forest and upward-forest.

#### 4.1.1 Single-Bounded: DAG

We begin with the auxiliary result that  $\text{wcase}(N)$  can be computed in time polynomial in the number of nodes in the graph. This result is used later to bound the complexity of Single-Bounded.



**Figure 3: Partition Example**

**THEOREM 4.2 (COMPUTATION OF WORST CASE).** *Given a node set  $N$  at which questions are asked,  $wcase(N)$  can be computed in  $O(n^2 \cdot k)$ , where  $n = |V|$ .*

A formal proof can be found in Appendix A.1. The main idea is to first compute all pairs  $(a, b)$  in  $V \times V$  such that  $b \in rset(a)$  and then use this information to compute  $cand(N, u^*)$  for every possibility of  $u^*$ . Using the result above, we can define a brute-force approach to solving Single-Bounded, by considering all possible combinations of  $N$  with size at most  $k$ .

**LEMMA 4.3 (BRUTE-FORCE SOLUTION).** *The optimal solution of Single-Bounded for any DAG can be found in  $O(n^k \cdot n^2 \cdot k)$ , where  $n$  is the number of nodes in  $V$ .*

Clearly, we can solve Single-Bounded optimally in PTIME if  $k$  is bounded by a constant. However, the appearance of  $k$  in the exponent hints at the hardness of the problem in the general case. Indeed, the following result shows that Single-Bounded is computationally hard. The proof of the following theorem, presented formally in Appendix A.3, shows a reduction to Single-Bounded from the NP-hard max-cover problem [14].

**THEOREM 4.4.** *Single-Bounded cannot be solved in polynomial time unless  $P = NP$ .*

Even though the general problem is intractable, it may be possible to find efficient solutions by leveraging specific characteristics of the input, and in particular of the DAG  $G$ . The following subsections examine this hypothesis for the two cases identified in the problem dimensions: downward-forests and upward-forests.

### 4.1.2 Single-Bounded: Downward-Forest

In the *downward-forest* case,  $G$  is a forest of directed trees with edges from parents to children nodes (see also Section 2.)

We begin by showing that Single-Bounded on a downward-forest can be reduced to Single-Bounded on a downward-tree (a tree with edges from parents to children), by attaching a virtual root node that links all the trees in the forest. The following theorem is proved by showing that the optimal solution for the resulting tree gives a solution to the forest with  $wcase$  at most one more than optimal.

**THEOREM 4.5 (DOWNWARD-FOREST  $\Rightarrow$  TREE).** *Given a downward-forest  $G_F$ , there exists a downward-tree  $G_T$ , such that a solution  $N_T$  to HumanGS on  $G_T$  gives a node set  $N'_T$  of  $G_F$  such that  $wcase(N'_T) \leq wcase(N) + 1$ , for any node set  $N$  of  $G_F$  where  $|N|, |N'_T| \leq k$ .*

We can therefore focus on solving the Single-Bounded problem for a single downward-tree, instead of a downward-forest, ignoring the additive constant of  $\leq 1$ . We show that this problem is equivalent to the *partition problem* [21], which admits efficient solutions.

**DEFINITION 4.6 (PARTITION PROBLEM).** *Given an undirected tree, find  $k$  edges such that their deletion minimizes the size of the largest connected component.*

To show the equivalence, we first define how a chosen set  $N$  induces a partition of the tree into connected components.

**DEFINITION 4.7 (PARTITION ON A NODE SET).** *In a downward-tree, we recursively define the partitions on a node set  $N$ , denoted  $P(N)$ , as the following:*

- If  $x \in N$  and none of  $x$ 's descendants are in  $N$ , then the subtree under  $x$  (including  $x$ ) is a partition. (We call this partition the partition of  $x$ .)
- If  $x \in N$  and some of  $x$ 's descendants are in  $N$ , then the subtree under  $x$  (including  $x$ ) excluding all of the partitions of  $x$ 's descendants is a partition. (We call this partition the partition of  $x$ .)
- Whatever is left after all the partitions are formed is a partition. (If  $x$  is the root of the remainder of the tree, then the partition is called the partition of  $x$ .)

Note that we have exactly  $|N|$  or  $|N|+1$  partitions. As an example, consider Figure 3. Here there are three partitions corresponding the node set  $N = \{y, z\}$ , i.e., the partition of  $x$ ,  $y$  and  $z$ .

**LEMMA 4.8 (CANDIDATE SET PARTITION).** *Given questions asked at a node set  $N$ , the candidate set  $cand(N, u^*)$  for any  $u^*$  corresponds to one of the partitions from  $P(N)$ .*

The proof of the lemma is given in Appendix A.5. The previous result essentially establishes the equivalence between the two problems, as our goal is to minimize  $wcase(N)$ , which is equivalent to the size of the largest partition that can be induced by  $N$ . Since the partition problem can be solved in PTIME, it follows directly that the same holds for Single-Bounded on a downward-tree. The following theorems formalize these observations.

**THEOREM 4.9 (PARTITION PROBLEM EQUIVALENCE).** *The problem of Single-Bounded on downward-trees is equivalent to the partition problem.*

Using a dynamic programming algorithm from [21] for the partition problem, we obtain the following result for Single-Bounded: (The proof of both these theorems is in the appendix.)

**THEOREM 4.10 (SINGLE-BOUNDED).** *There exists an algorithm with complexity  $O(n \log n)$  that solves Single-Bounded on a downward-forest.*

### The Case of Balanced Trees

We also examine a specialization of the Single-Bounded problem on *balanced* downward-trees in Appendix A.8. Balanced trees may arise in specific applications, and we show that solutions can be computed for them much more efficiently than for arbitrary trees. Additionally, the solution to balanced trees may provide us with intuition on how to handle nearly balanced trees that may arise in concept hierarchies or phylogenetic trees.

### 4.1.3 Single-Bounded: Upward-Forest

Next, we assume that  $G$  is an upward-forest. An upward-forest is a collection of upward-trees, where an upward-tree is a directed tree with directed edges from the children nodes to the parent.

We begin with a theorem that shows that it is sufficient to study upward-trees instead of upward-forests, by augmenting the forest with a new virtual root node such that there is an edge from each root node of each of the upward-trees to the new root node. The proof of the following result, sketched in Appendix A.9, is similar to that of Theorem 4.5. Note that the additive factor is increased by 1, since asking a question at the root of an upward-tree (unlike the root of a downward-tree) can change the size of  $wcase$  by 1.

**THEOREM 4.11 (UPWARD-FOREST  $\Rightarrow$  TREE).** *Given an upward-forest  $G_F$ , there exists a upward-tree  $G_T$ , such that a solution  $N_T$  to HumanGS on  $G_T$  gives a node set  $N'_T$  of  $G_F$  such that  $wcase(N'_T) \leq wcase(N) + 2$ , for any node set  $N$  of  $G_F$  where  $|N'_T|, |N| \leq k$ .*

Therefore, instead of upward-forests, we consider upward-trees. We visualize the tree with the root node at the top (with all edges going upward). This way, if we say a node  $x$  has two children  $y$  and  $z$ , we are disregarding the direction of the edges, which go from nodes  $y$  and  $z$  to  $x$ .

We use a dynamic programming algorithm, listed in Algorithm 2 in the appendix. Intuitively, the algorithm collects all possible worst-case contributions to the candidate set for the subtree under a certain node, and then combines the worst-cases when going from children to parent. For instance, the contribution to the overall candidate set when  $x$ 's subtree does not contain the target node is nothing but the sum of the contributions from the children of  $x$ , when neither of them have the target node in their subtree.

After the bottom-up process, at the root, the algorithm picks the tuple for which the overall worst-case candidate set is the smallest, and traces it all the way to the leaves to give the optimal solution. The following theorem establishes the correctness and complexity of the algorithm. The detailed proof of the theorem can be found in the Appendix A.10.

**THEOREM 4.12 (UPWARD-FOREST).** *There exists an algorithm that solves the Single-Bounded problem for  $k$  questions in  $O(m \cdot k^2 \cdot n^5)$  on an  $m$ -ary upward-forest with  $n$  nodes.*

### The Case of Balanced Trees

Similar to our discussion for downward-trees, we consider the specialization for balanced upward-trees in Appendix A.11. For balanced upward-trees, we solve Single-Bounded in  $O(1)$ .

## 4.2 Single-Unlimited

In the Single-Unlimited problem, we do not have a strict budget on the number of questions that can be asked; instead, we want to find the smallest set of questions  $N$  such that the target node is uniquely determined in the every case.

**DEFINITION 4.13 (SINGLE-UNLIMITED).** *(Unlimited Search for a single target node) Given that the target set  $|U^*| = 1$ , find the smallest set  $N \subseteq V$  to ask questions such that  $wcase(N) = 1$ .*

First, we illustrate in the following example that the number of questions that are required to ensure that  $wcase(N) = 1$  can vary widely depending on the structure of the underlying graph  $G$ . Subsequently, we study various structures of  $G$  and explore algorithms to solve Single-Unlimited.

**EXAMPLE 4.14.** *There exist connected graphs for which the smallest node-set  $N$  is almost all the nodes. Consider a downward-tree with one parent and  $n$  immediate children. Suppose that we leave two of the  $n$  children unasked, and that one of these is the target node. Then the unasked node will still be present in the candidate set. Hence, we need to ask  $n - 1$  nodes in this scenario.*

*Also, there are graphs for which  $|N|$  is  $O(\log n)$ . Consider a graph with nodes in two levels,  $r$  nodes in the first level, and  $2^r$  nodes in the second level. We add directed edges from the nodes in the first level to those in the second level, as follows: There is an edge from the  $j$ -th node in the first level to the  $i$ -th node in the second level (always counting from the left) if and only if there is a 1 in the  $j$ -th position of the binary encoding of  $i$ .*

*In this case, if we ask all  $r$  nodes in the first level, we only need to ask at most  $r$  additional nodes in the second level (precisely those corresponding to  $1, 2, 4, \dots, 2^{r-1}$ , since each of these  $r$  additional nodes in the second level has a single parent, and so if the parent returns YES, then it is unclear if the node or its parent is the target node). All other nodes in the second level have a unique encoding with respect to the  $r$  nodes in the first level. Thus the answers*

*returned by the  $r$  parent nodes are sufficient to infer whether or not one of these nodes in the second level is the target node. Thus, the size of  $|N| = O(\log n)$  for this graph.*

### 4.2.1 Single-Unlimited: DAG

Our first result is a general scheme to solve Single-Unlimited for any  $G$  using our results from Single-Bounded: By repeating Single-Bounded with various values of  $k$ , it is possible to derive a solution for Single-Unlimited, formalized in the theorem below.

**THEOREM 4.15 (REPEATING SINGLE-BOUNDED).** *If there is an algorithm that solves Single-Bounded on a DAG in time  $T$ , then Single-Unlimited on the same DAG can be solved in  $O(K \times T)$ , with  $K = \min\{\log^2 o, \log n\}$ , where  $o = \text{size of the optimal } N$ .*

The approach is to run Single-Bounded repeatedly using either binary search on  $n$ , or by doubling  $k$  until we locate the smallest  $N$  for which  $wcase = 1$ . The proof can be found in Appendix B.1.

### 4.2.2 Single-Unlimited: Downward-Forest

On a downward-forest, Single-Unlimited can be solved in  $O(n)$ :

**THEOREM 4.16 (DOWNWARD-FOREST).** *On a downward-forest, we need to ask almost all nodes (except at most one) in order to solve Single-Unlimited.*

The proof may be found in Appendix B.2. The main insight in the proof is that if we leave a node unasked, then on getting a NO answer from all the children of the node and a YES answer from the parent of the node, we have  $wcase > 1$  because we cannot distinguish between the node and its parent. However, in the general case of several trees in the forest, one of the roots need not be asked. This is because there is a single target node, and if all children of this root as well as everything else returns NO, then this root has to be the target node.

### 4.2.3 Single-Unlimited: Upward-Forest

The following theorem presents our result on Single-Unlimited on an upward-forest. The formal proof is in the appendix, but the main intuition is that we do not need to ask questions at any internal node with degree  $> 1$ , once all leaves have been asked. Note that the answer returned by every such internal node can be inferred from the answers of the children, bottom up. (If all the children return YES, then the node returns YES, else it returns NO.) Also, if we leave more than one leaf unasked, then there could be ambiguity regarding which of them is the target node.

**THEOREM 4.17 (UPWARD-FOREST).** *In order to solve Single-Unlimited on an upward-forest, we need to ask questions at all the leaves (except at most one), and all internal nodes with indegree 1, as in Algorithm 4 in the appendix.*

From the theorem we see that unlike the downward-forest case, we now do not need to ask questions at almost all nodes. The pseudocode is listed in Algorithm 4 in the appendix.

## 5. MULTIPLE TARGET NODES

In the Multi version of the problem, there exists a target set  $U^* \subseteq V$ , which denotes the unknown set of nodes we wish to discover by asking questions. Unlike the Single case, the size of the target set can be any  $|U^*| \geq 1$  and is unknown. The only constraint we are given is that the target nodes satisfy the independence property, i.e., no two nodes in  $U^*$  are related. We impose this constraint based on our motivating applications. (See Section 2.)

Computation of the candidate set for the Multi problem can be found in Section 3. To recap, a single question at  $u$  lets us exclude from the candidate set either  $\text{pset}(u)$  (if the answer is YES) or  $\text{rset}(u)$  (if the answer is NO).

To incorporate the independence property in defining a worst-case candidate set, we define the function  $ip$  on a set of nodes to return *true* if and only if the set of nodes satisfy the independence property. Given a set  $N$  of nodes, we redefine the *Worst-Case Candidate Set* to be:

$$\text{wcase}(N) = \max_{U \subseteq V, ip(U)=\text{true}} |\text{cand}(N, U)|$$

Next we study the Bounded (Section 5.1) and Unlimited (Section 5.2) versions of the HumanGS problem for the Multi case.

## 5.1 Multi-Bounded

In this section we consider the bounded search problem for a target set of nodes, formally stated below.

**DEFINITION 5.1 (MULTI-BOUNDED).** (*Bounded Search for a target set.*) Given a parameter  $k$ , find a set  $N$  of nodes  $N \subseteq V, |N| = k$  to ask questions such that  $\text{wcase}(N)$  is minimized.

We present algorithms and complexity results for Multi-Bounded, examining various properties of the structure of the graph  $G$ . Section 5.1.1 considers arbitrary DAGs, Section 5.1.2 considers downward and upward-forests.

### 5.1.1 Multi-Bounded: DAG

In this section, we establish the overall complexity of Multi-Bounded for an arbitrary DAG. We first show an NP-hard lower bound, and follow it with an upper bound of  $\Sigma_2^P$ .<sup>2</sup> Bridging the gap between our lower and upper bounds is an open problem.

The following theorem establishes the NP-hardness of Multi-Bounded. As in Single-Bounded, we use the max-cover problem to prove NP-hardness, although the details of our reduction need to be modified for the Multi-Bounded problem. The proof of hardness can be found in Appendix C.1.

**THEOREM 5.2 (LOWERBOUND).** *The Multi-Bounded problem is NP-Hard in  $n$  and  $k$ .*

The following theorem whose proof can be found in Appendix C.2 establishes the upperbound on the complexity of Multi-Bounded.

**THEOREM 5.3 (UPPERBOUND).** *The decision version of Multi-Bounded<sup>3</sup> is in  $\Sigma_2^P$ .*

### 5.1.2 Multi-Bounded: Downward/Upward-Forests

Next we consider Multi-Bounded for forests of arbitrary trees. We can extend Theorems 4.5 and 4.11 to the case of Multi-Bounded, which enables us to focus our attention on trees instead of forests. Our first result shows that we do not need to consider upward and downward-trees separately. We then present the main result of this section, providing a PTIME dynamic programming algorithm that solves Multi-Bounded for downward-trees (and thereby downward and upward-forests).

**THEOREM 5.4 (EQUIVALENCE).** *Multi-Bounded on a downward-tree is equivalent to Multi-Bounded on an upward-tree.*

The proof can be found in Appendix C.3. Intuitively, if we reverse all the arrows, and we complement all the answers (replace YES

<sup>2</sup> $\Sigma_2^P$  corresponds to a class of problems higher than P and NP.

<sup>3</sup>The decision version of Multi-Bounded is defined as follows: We want to test whether  $\text{wcase}(N)$  is reduced to a size below some  $\tau$ .

with NO and NO with YES), we can transform Multi-Bounded on a downward-tree to that on an upward-tree and vice versa.

We use a dynamic programming algorithm to solve Multi-Bounded on a upward/downward-forest (Algorithm 5 in the appendix). This algorithm is similar to the one used to solve the Single-Bounded on an upward-forest. However, there are some differences, as we will see. Since downward and upward-trees are equivalent, we describe our algorithm on a downward-tree. The main insight in the algorithm is that if we solve the sub-problem of finding the worst-possible contributions to the overall candidate set while allocating  $k$  questions to a node, then these contributions can be combined bottom-up to solve the problem at the parent.

For each node  $x$ , for each  $i \in \{0, \dots, k\}$ , we generate and populate array  $T_x[i]$ , which contains a set of worst-case contributions corresponding to each allocation of  $i$  questions in total to  $x$  and the two children  $y$  and  $z$ . Each scenario is a 4-tuple,  $((k_1, k_2), p_1, p_2, n)$  where the first entry denotes that this 4-tuple arises from an allocation of  $k_1$  questions to the sub-tree under  $y$  and  $k_2$  questions to the sub-tree under  $z$ . For this particular allocation of questions,  $p_1$  is the contribution to the candidate set such that (a) it contains  $x$  and (b) there is a target node below  $x$ ,  $p_2$  is the contribution such that (a) it does not contain  $x$  and (b) there is a target node below  $x$ , and  $n$  is the contribution if there is no target node below  $x$ . Note that there is no  $r$  value in each tuple (unlike Single-Bounded) since the presence (or not) of a target node above  $x$  in the tree will not change the answers to questions in  $x$ 's subtree if we already know that  $x$ 's subtree does/does not contain a target.

**THEOREM 5.5 (DP ALGORITHM).** *There exists an algorithm that solves the Multi-Bounded problem for forests in  $O(k^2 \cdot n^6)$ .*

The details and proof of correctness of the theorem above can be found in Appendix C.4.

### The Case of Balanced Trees and Forests

Multi-Bounded on a balanced tree can be solved in  $O(1)$  by picking nodes from a certain level as described in Appendix C.5. Additionally, we can combine results for balanced trees using a dynamic programming algorithm to solve the problem for a balanced forest (i.e., a forest of balanced trees) in  $O(k^2 \cdot r)$ , where  $r$  is the number of balanced trees. Details can be found in Appendix C.6.

## 5.2 Multi-Unlimited

Finally, we address the problem of unlimited search:

**DEFINITION 5.6 (MULTI-UNLIMITED).** (*Unlimited Search in a DAG for a target set*) Find the smallest set of nodes  $N \subseteq V$  to ask questions such that  $\forall U^* \subseteq V$  satisfying  $ip(U^*) = 1$ , we have  $|\text{cand}(N, U^*)| = |U^*|$ .

The following theorem shows an interesting result that the Multi-Unlimited problem is “trivialized” by the fact that questions need to be asked at all nodes to ensure that no extraneous nodes remain in the candidate set.

**THEOREM 5.7 (TRIVIALITY).** *The optimal solution to an instance of Multi-Unlimited is  $N = V$ , i.e., we need to ask a question at every node in the graph.*

Intuitively, we need to ask a question at every node  $x$  because if we get a YES response from all of  $x$ 's ancestors, and a NO response from all of  $x$ 's descendants, we cannot be sure if  $x$  is in the target set or not. For a formal proof, refer Appendix D.

## 6. EXPERIMENTAL STUDY

We conducted an experimental study of our HumanGS algorithms using a webpage categorization task on the real-world DMOZ

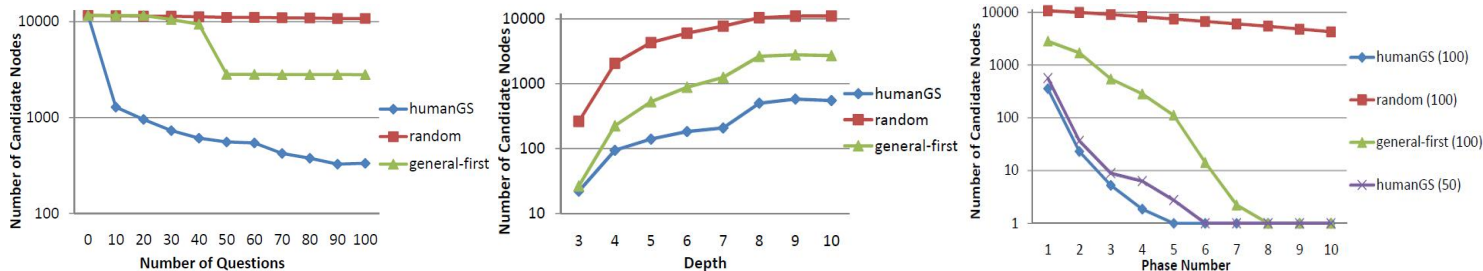


Figure 4: Experiments on (a) varying number of questions asked (b) varying the size of the tree (c) varying the number of phases

concept hierarchy (<http://dmoz.org>). DMOZ is a human-curated internet directory based on a downward tree of categories. The goal is to assign websites of interest to nodes in this downward tree. In general, human judgement is needed for this assignment.

To expedite the process of assigning new web-pages to categories, we use the algorithms developed for HumanGS to select which questions to ask humans. A question at the node corresponding to category X would be of the form: does this new webpage fall under X? Note that questions for a given web-page can be asked concurrently to independent humans, and the answers can then be combined to determine the candidate set of categories.

**Experimental Objectives.** Our experimental study has two important features that complement the analysis presented in previous sections. First, we study how our algorithms behave on average (rather than worst case) for a specific problem over real data. Second, since the algorithm for the Single-Unlimited problem for a downward tree is impractical, we study the performance when we use many iterations of the Single-Bounded algorithm. We now describe these features in more detail.

Recall that our algorithms are provably optimal in terms of minimizing the worst-case size of the candidate set. In our experiments, we would like to complement that theoretical analysis with measurements of the actual size of the candidate set after obtaining answers to the questions selected by the algorithm. By measuring the actual size, our goal is to examine whether the worst-case objective function also corresponds to good *average-case* performance.

One algorithm we could use for the webpage categorization task is Single-Unlimited. This algorithm selects questions offline to guarantee a wcase of size 1, but requires asking questions at almost every node, which is clearly impractical. Instead, since the average case may not correspond to the worst case, we would like to see how much the candidate set can be reduced by asking a bounded number of questions (i.e., using the algorithms for Single-Bounded). Using the answers to these questions, we may be able to precisely determine the target node. However, if the target node is not precisely determined, we may need to invoke the algorithm once again to issue another bounded set of questions to the crowdsourcing service. Thus, a practical method of using our algorithm is in *phases* where in each phase we invoke the Single-Bounded algorithm in order to select  $k$  additional questions by taking the current candidate set as the input downward tree. The net effect is that each phase shrinks the candidate set further, until the target node is identified or the candidate set is small enough to assign to a human worker for the final solution. Note that this approach is a hybrid approach between a completely offline approach and a completely online approach (when we issue a single question at a time).

We present experiments to evaluate the average-case performance of our algorithms, focusing on the following questions:

- How does the candidate set shrink
  - as we vary the number of questions asked?
  - as we vary the size of the input downward tree?
  - when multiple phases are used?

- What is the relationship between the number of questions per phase and the total number of phases in order to solve the HumanGS task?

## 6.1 Methodology

**Task Specifics.** We evaluate our algorithms with a webpage categorization task on the science sub-tree of the DMOZ hierarchy (containing over 11,600 nodes). Each task is the placement of a web page into the hierarchy. We simulate each task by picking a node in the hierarchy where the web page would go. Then we simulate the crowd-sourcing service by answering the questions asked by our algorithms truthfully. If an actual crowd-sourcing service is used, correctness may be ensured by having multiple humans attempt the same question.

**Tested Algorithms.** We implemented our algorithm for Single-Bounded for a downward tree. We henceforth refer to this algorithm as *humanGS*. We compare *humanGS* against two baseline algorithms. The first algorithm, *random*, simply picks  $k$  random nodes from the downward tree. The second algorithm, termed *general-first*, asks questions at the first  $k$  nodes encountered in a breadth-first traversal starting from (but excluding) the root.

Recall that each algorithm is executed in phases, where each phase receives as input the graph formed from the candidate set computed using all previous phases.

**Metrics.** We measure the performance of an algorithm as the actual size of the candidate set after asking the questions selected by the algorithm. To ensure statistical robustness, we test each algorithm on a set of 100 random tasks, each generated by sampling the target node uniformly at random from the nodes of the input tree, and we report the average size of the candidate set over all tasks in the test set. For algorithm *random*, we perform an additional averaging step over 10 runs of the algorithm per test task, in order to mitigate the effects of random question sampling.

## 6.2 Results

**Effect of Number of Questions.** This experiment examines how the candidate set shrinks on increasing the number of asked questions in one phase. We use  $k$  to denote the number of questions, and focus on the first phase where the input is the complete hierarchy.

Figure 4(a) shows the average number of candidate nodes as we vary  $k$ . Note that the y-axis is in log scale. As shown, our *humanGS* algorithm outperforms the baseline algorithms by an order of magnitude for all tested values of  $k$ . In particular, *humanGS* reduces the candidate set to around 1000 nodes with just 10 questions (a 10-fold decrease), whereas *random* and *general-first* flatten out well above 1000 nodes even after 100 questions. The steep drop for *general-first* at around 45 questions occurs because in our dataset there is a node close to the root with a large number of descendants. When we select that node, a large part of the hierarchy is pruned.

**Effect of Tree Size.** The second experiment examines the performance of the three algorithms as we vary the size of the input tree. We varied the size of the downward tree by restricting the depth of the tree, i.e., any node at a greater depth is removed. The target node is sampled from the remaining nodes. We set  $k = 50$  and

focus again on a single phase.

Figure 4(b) depicts the average candidate size (again in log scale on the y-axis) against the depth of the tree. Algorithm *humanGS* continues to outperform its competitors, with an increasing margin as we approach the actual size of the tree. Its performance is matched by *general-first* only for a “trivial” depth of three, which corresponds to an unrealistically small task.

**Phase-based Operation.** The final experiment evaluates the overall performance of the phase-based approach. Our goal is to examine how fast each algorithm identifies the single target node of the specific HumanGS task.

Figure 4(c) depicts the average candidate-set size for the three algorithms as we increase the number of phases, when  $k = 100$  for each phase, once again with the y-axis on a log scale. The three plots corresponding to 100 questions per phase are denoted *humanGS (100)*, *random (100)* and *general-first (100)* in the figure. (We also plot the curve for *humanGS* when  $k = 50$  and we discuss it in the next paragraph – depicted as *humanGS (50)* in the figure.) We observe that *humanGS* yields the fastest decrease among the three algorithms. For instance, *humanGS* is able to identify the target node after 5 phases on average. This compares favorably to the eight phases required by *general-first*, whereas *random* was not able to decrease the candidate set below 10000 nodes for the whole experiment. Additionally, the candidate-set size of *humanGS* is below 20 after only two phases. In contrast, the corresponding sizes for *general-first* and *random* are close to 1000 and 10000 respectively. Note that when the candidate set size is small, we may also consider giving the candidate nodes to a human worker in order to make the categorization.

Focusing on the two curves for *humanGS*, we observe a small degradation in performance from  $k = 100$  to  $k = 50$  for the same number of phases. However,  $k = 50$  yields much better overall performance if we consider the total number of questions asked. For instance, with two phases and  $k = 50$  (a total of 100 questions), *humanGS* performs an order of magnitude better than having a single phase with  $k = 100$ . Furthermore, *humanGS* requires 6 phases to discover the single target node with  $k = 50$  (a total of 300 questions), compared to 5 phases for  $k = 100$  (a total of 500 questions). These results indicate that increasing the number of phases is more beneficial compared to increasing the number of questions per phase. The trade-off of course is in latency, since the extra phases mean additional round-trips to the crowd-sourcing service, and reduced parallelism. Examining this trade-off in more detail is an interesting direction for future work.

## 7. RELATED WORK

There have been several studies of the social aspects of crowd-sourcing technologies and on how to design social games (e.g., captchas and GWAP — Games With A Purpose) in order to accomplish tasks [4, 12, 13]. Previous works have also examined the use of crowd-sourcing technologies to accomplish specific tasks, such as natural language annotations [18], video and image annotations [11, 6], and search relevance [16]. However, none of these studies consider the problem of minimizing the set of questions to ask users; additionally, the tasks studied are fairly simple and unstructured, unlike HumanGS which (a) utilizes the inherent structure of a DAG to ask questions, and (b) requires asking multiple questions in order to be “solved” correctly.

The field of active learning [23] also studies the problem of requesting human input. The goal is to request input from experts with the maximal “information content”/ informativeness, which is similar in spirit to our problem. However, this human input is only used to generate training data for machine learning tasks (es-

pecially when the current training data is insufficient or not informative). Additionally, most work in active learning does not ask questions to humans in parallel, and thus does not leverage the inherent parallelism in crowd-sourcing systems.

There has been some work on the utilization of human input in data integration and exploration [20, 25, 22]. Similar to HumanGS, the aim is to minimize the number of posed questions in order to minimize the total cost. However, the models and applications studied are very different from ours, and not as generally applicable. In addition, the tasks are such that the human inputs “assist” the computer program. In our case, the tasks are fairly complex, as a result of which they require primarily human input to solve them.

Our problem is similar to that of decision trees [7]. In decision trees, we wish to classify a target item as belonging to one of many classes (in our case, classify a target item as being one of the nodes in the DAG). However, unlike decision trees, we do not have a training set (or statistics of various classes) and there are no attributes on which a classifier can be built. The only questions that we can ask are those that involve reachability, and the optimization issues that arise are very different from those that arise in the construction of a decision tree.

There are other domains with optimization problems that involve searching with a budget, e.g., searching in a 2-dimensional area for oil [5], where the aim is to find oil as quickly as possible by selecting areas to drill and the depths of drilling. There has also been work on multi-armed bandits [10] and similar stochastic models for searching for an optimal solution that yields most revenue in computation advertising and machine learning settings. These models analyze the tradeoffs between *exploration vs. exploitation*. However, the searching model in these domains is very different from ours, and hence the solutions proposed are different as well.

## 8. CONCLUSIONS AND FUTURE WORK

In this work, we presented HumanGS, a general problem that arises in many problems of human computation. We explored the problem space via three orthogonal axes: Single / Multi, Bounded / Unlimited and DAG / Downward-Forest / Upward-Forest, and developed algorithms for all combinations. Our algorithms generate the optimal set of questions that can be asked to humans. These questions do not rely on the answers received for prior questions, and can be issued in parallel in a crowd-sourcing system.

Since optimization of human computation is an unexplored area, there are many interesting avenues of future work. Within HumanGS itself, one interesting direction is to assume that we are given a probability distribution on the selection of target nodes, which we can use to pick questions in order to minimize the expected size of the candidate set. Additionally, some questions asked to humans may be answered incorrectly, and in such a case we would like to perform error-resilient HumanGS. Last, but not the least, we would like to explore other problems in the human computation space that require optimization by way of picking the optimal questions to ask humans.

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## APPENDIX

### Outline

We cover the details of Single-Bounded in Appendix A, Single-Unlimited in B, Multi-Bounded in C and Multi-Unlimited in D.

In the appendix, we provide

- the complete analysis of Multi-Unlimited, and
- the analysis of balanced trees for Single and Multi-Bounded

as well as:

- proofs for some theorems and
- pseudocode for the algorithms

from the main body of the paper.

## A. Single-Bounded

### A.1 Proof of Theorem 4.2

PROOF. In time  $O(n^2)$ , for all pairs of nodes  $a$  and  $b$  in  $G$ , we can compute and maintain whether or not  $a \in \text{rset}(b)$ . Let  $u^*$  be the target node in the graph. In time  $O(|N|)$ , we can compute the answers to each of the questions asked at the nodes in  $N$ . Subsequently, we can compute  $\text{cand}(N, u^*)$  in  $O(n \cdot |N|)$  by checking the following for every node  $u$ : if there is a node in  $u' \in \text{rset}(u)$ ,  $u \neq u'$  such that  $u' \in N$  and  $u'$  returned YES, or if there is a node  $u \in \text{pset}(u)$ , such that  $u \in N$  and  $u'$  returned NO, then  $u \notin \text{cand}(N, u^*)$ , else  $u \in \text{cand}(N, u^*)$ . We can compute  $\text{wcase}$  by repeating the above procedure for every possible  $u^*$ , taking a total of  $O(n^2 \cdot |N|)$  time.  $\square$

### A.2 Proof of Lemma 4.3

PROOF. We find  $\text{wcase}$  for each choice of  $\binom{n}{k}$  nodes at which questions are asked. The choice for which the worst-case candidate set is the smallest is the optimal solution.  $\square$

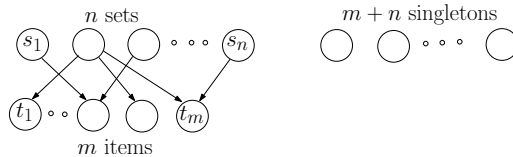


Figure 5: Hardness Proof for Single-Bounded

### A.3 Proof of Theorem 4.4

PROOF. We prove that the decision version of the problem is NP-complete, stated as follows: Given a budget  $k$  and a positive integer  $m$ , is there a node-set  $N$  to ask questions such that  $\text{wcase}(N) \leq m$ ? We refer to this problem as Single-Bounded-DECISION.

We use a reduction from the NP-complete *max-cover problem* [14]. In this problem, the objective is to pick a certain number of sets such that they cover as many items as possible. (A set containing a given item is said to cover that item.) Let there be  $m$  items and  $n$  sets in the max cover problem. We need to select  $k$  sets such that the maximum number of items are covered.

We reduce the max-cover problem to Single-Bounded-DECISION with the following directed acyclic graph. (An instance of the graph is shown in Figure 5.) Consider nodes arranged in two layers. In the first layer, we have one node corresponding to each set  $s_i$  in the max cover problem. In the second layer, we have a node corresponding to each item  $t_i$  in the max-cover problem. There is a directed edge from the node corresponding to set  $s_i$  to the node corresponding to the item  $t_j$  iff item  $t_j$  is present in set  $s_i$ . In addition, we include  $n+m$  singleton nodes (nodes with no incoming or outgoing edges) in the DAG. Subsequently, we call Single-Bounded on this DAG with a budget of  $k$  questions. The worst-case candidate set corresponds to each of the questions in  $N$  receiving NO answers. To see this, note that if we get a YES answer to a question asked at any of the nodes corresponding to sets, the candidate set is  $\leq m$ . Receiving a YES answer to any question asked at the nodes corresponding to items or singletons will give a candidate set of 1. However, if we get all NO answers, the number of nodes remaining in the worst-case candidate set is at least  $m+n$ , even if all the nodes corresponding to items as well as  $k$  from the nodes corresponding to sets and singletons are eliminated from the candidate set. Additionally, the solution of the Single-Bounded problem, i.e.,  $N$ , will only contain nodes corresponding to sets, because those nodes exclude the maximum number of nodes from the candidate set, in the worst case. Thus, Single-Bounded picks nodes corresponding to sets such that the maximum number of nodes corresponding to items are covered. Therefore, the solution to Single-Bounded corresponds to a max-cover. Conversely, every solution of the max-cover problem can be written as a solution for Single-Bounded.

In addition, given that we can compute  $\text{wcase}(N)$  in PTIME (see Theorem 4.2), a solution for Single-Bounded-DECISION can be verified in PTIME. Thus, Single-Bounded-DECISION on DAGs is in NP and is NP-Complete.  $\square$

### A.4 Proof of Theorem 4.5

PROOF. We augment the downward-forest with a single root node such that there exists an edge from the new root node to the root of each of the trees in the directed forest. Let the original forest be  $F$  and the new augmented tree (a downward-tree) be  $T$ .

Let  $N_T$  be an optimal selection of  $k$  nodes from  $T$  at which questions are asked. We first convert  $N_T$  into  $N'_T$  such that there is no question asked at the root. (We delete the root from  $N_T$ , if present.) Since a question at the root has to return a YES answer, the worst-case candidate set at  $T$ , denoted by  $\text{wcase}_T$ , will be unchanged. We therefore have  $\text{wcase}_T(N'_T) = \text{wcase}_T(N_T)$ .

Let  $N_F$  be the optimal solution on  $F$ . If we select set  $N_F$  as the questions to be asked at  $T$ , we get  $\text{wcase}_T(N_T) \leq \text{wcase}_T(N_F)$ ,

since  $N_T$  gives the optimal worst-case candidate set for  $T$ . Also, since there is an addition of at most one node to the worst-case candidate set when  $N_F$  is used on  $T$  instead of  $F$ ,  $\text{wcase}_T(N_F) \leq \text{wcase}_F(N_F) + 1$ , where  $\text{wcase}_F$  is the worst case candidate set when the questions are asked at the nodes in  $F$ . Now, apply  $N'_T$  on  $F$ . We then have  $\text{wcase}_F(N'_T) \leq \text{wcase}_T(N'_T) = \text{wcase}_T(N_T) \leq \text{wcase}_F(N_F) + 1$ . Thus, choosing the questions to minimize worst-case candidate set in the tree gives the optimal worst-case candidate set for the forest plus at most one more node. We ignore the additive factor of 1 in our calculations.  $\square$

## A.5 Proof of Lemma 4.8

PROOF. If  $u^* \in$  partition  $p$  of  $x$ , then we prove that  $\text{cand}(N, u^*) = p$ . First, we consider the case when  $x \in N$ . In this case, notice that the question at  $x$  returns a YES answer. Using Theorem 3.3, the candidate set can be restricted to the subtree under  $x$ . The only nodes  $\in N$  at which the answer is YES are those that are ancestors of  $x$ , but they do not change the candidate set. Those nodes  $\in N$  that are descendants of  $x$  or unrelated to  $x$  all return NO, since there is no path from those nodes to the target node  $u^*$ . Since all those nodes return NO, the subtrees under those nodes can be removed from the candidate set. Thus, the candidate set is precisely  $p$ .

If  $x \notin N$ , then  $x$  is the root of the directed tree. Here, once again, if we asked a question at  $x$ , we would obtain a YES answer. Those nodes  $\in N$  that are descendants of  $x$  all return NO, since there is no path from those nodes to the target node  $u^*$ . Since all those nodes return NO, the subtrees under those nodes can be removed from the candidate set. Thus, the candidate set is once again  $p$ .  $\square$

## A.6 Proof of Theorem 4.9

PROOF. As seen in Lemma 4.8, the candidate set on asking a question at any node corresponds to one of the partitions induced by asking questions. Thus, in order to minimize  $\text{wcase}$ , it is sufficient to solve the partition problem on the downward-tree. The size of the largest partition is the size of the worst possible candidate set. Conversely, every instance of the partition problem can be cast as an instance of the Single-Bounded problem for a downward-tree. Thus, the two problems are equivalent.  $\square$

## A.7 Proof of Theorem 4.10

PROOF. The algorithm solves the equivalent partition problem on the downward-tree formed from the downward-forest augmented with a root node. Reference [21] gives a dynamic programming algorithm that finds the minimal number of edges that need to be cut in order to achieve a certain partition size in  $O(n)$ . We run binary search over partition sizes in order to find the smallest partition size for which the minimal number of edges to be cut is  $\leq k$ .  $\square$

## A.8 Balanced Downward-Trees

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### Algorithm 1: Single-Bounded Balanced Downward-Tree

---

**Data:**  $\mathcal{G} = m$ -ary balanced downward-tree with depth  $d, k$

**Result:** optimal set of  $k$  nodes to ask questions to

$l := \lfloor \log_m k \rfloor$ ;

$N :=$  all nodes at depth  $l$ ;

**if**  $k \neq m^l$  **then**

**for** all nodes  $x$  at depth  $l$  **do**

$N := N \cup$  first  $\lfloor (k - m^l)/m^l \rfloor$  children of  $x$ ;

**return**  $N$ ;

---

For a single balanced tree, the algorithm is listed in Algorithm 1 when  $k < m^{d/2}$ . Intuitively, the algorithm first picks all nodes at

depth  $l$ , where  $l$  is the maximum such depth that can be “blanketed” with questions. Once all nodes at this depth is added to  $N$ , the algorithm then proceeds to sub-divide each of the subtrees at depth  $l$  (i.e., those whose roots are at depth  $l$ ) by asking further questions, which are equally distributed among the subtrees at depth  $l$ . This approach effectively achieves a partition of the balanced tree by picking nodes from a depth such that the subtree under that depth is roughly of size  $1/k$  of the total size of the balanced tree.

**THEOREM A.1 (BALANCED DOWNWARD-TREE).** *For the case of a balanced downward  $m$ -ary tree of depth  $d$ , Single-Bounded can be solved optimally using Algorithm 1 in  $O(1)$  if  $k < m^{d/2}$ .*

PROOF. Consider a single balanced tree. Let  $m$  denote the degree of the tree and  $d$  its depth. (The root is at depth 0.) Suppose our budget is  $k = m^l$ . In such a situation, we ask questions at all nodes at depth  $l$ . First, let us compute the size of  $\text{wcase}$ . Since there is a single target node, there are only two possible outcomes for the corresponding answers: either all are NO because  $u^*$  is one of the  $(m^l - 1)/(m - 1)$  ancestors of the asked nodes, or exactly one of them is YES because  $u^*$  is one of the asked nodes or one of the  $(m^{d-l} - 1)/(m - 1)$  descendants of some asked node. Hence, the worst-case size of the candidate set is  $(\max\{m^l, m^{d-l}\} - 1)/(m - 1)$ . Given that  $m^l \leq k \ll n$ , we expect that  $m^l < m^{d-l}$  and hence the worst case is that the target node is a descendant of some node at depth  $l$ , and  $\text{wcase} = (m^{d-l} - 1)/(m - 1) = S$ .

To see that this is optimal, consider an alternate optimal solution  $N'$ . Now, consider all the sub-trees at depth  $l$ . Note that each subtree is of size  $S$ . If even one of the sub-trees  $t$  is unasked, i.e.,  $N \cap t = \emptyset$ , then by assigning all the ancestors of  $t$  present in  $N$  a YES answer, and all other nodes in  $N$  a NO answer, we get a worst-case candidate set larger than or equal to  $S$ . Also notice that there can be at most one question in each subtree at depth  $l$  (otherwise, there is a sub-tree that is unasked). Even if there is one subtree at depth  $l$  for which the root is in  $N$ , then the worst case does not change (assign YES to that node and to its ancestors in  $N$ , and NO to everything else in  $N$ ). Now consider the case when each sub-tree at depth  $l$  has a single question at depth  $> l$ . Here, if we assign all answers as NO, we get a candidate set  $> mS - m(S - 1)/m = (m - 1)S + 1 > S$ . Thus we have a contradiction, and the optimal solution is indeed the one that asks all nodes at depth  $l$ .

When  $m^l < k < m^{l+1}$ , we ask all nodes at depth  $l$ , then proceed to apportion equally the remaining questions to each of the subtrees at depth  $l$ . The remaining questions will be used at nodes at depth  $l + 1$ . For instance, let  $k = (x + 1)m^l + y$ , where  $x, y$  are integers and  $y < m^l$ . Here, we select all nodes at depth  $l$ , and  $x$  nodes from depth  $l + 1$  from each of the subtrees under the nodes at depth  $l$ . The worst-case candidate set is  $R = (m - x)(S - 1)/m + 1$ . Note that when  $x = m - 1$ , we find that the worst-case candidate set is  $(S - 1)/m + 1$ , which is one larger than  $(S - 1)/m$ , which would be the case if we ask all nodes at depth  $l + 1$ .

Now, to see that this approach is correct, let  $k = (x + 1)m^l$  (We show later that adding  $y < m^l$  extra questions does not affect  $\text{wcase}$ .) First, let there be a question asked at every node at depth  $l$ . (We show later that this is necessary.) In this case, we cannot get a smaller  $\text{wcase}$  on choosing a different  $N$  because moving any question to a depth  $> l + 1$  only makes the  $\text{wcase} > R$ . Also notice that if  $< x$  questions were allocated to any tree, then the  $\text{wcase}$  can only be larger, since  $m - x$  of the subtrees at depth  $l + 1$  would be unasked.

Now, to see that we need to ask every node at depth  $l$ , consider the following. If any node at depth  $l$  is asked, then at least  $x$  questions need to be allocated to the subtree under that node, and the worst case is at least as bad as  $R$ . If we were to not ask a question

at a node at depth  $l$ , then we need to ask at least  $x + 1$  questions at depth  $l + 1$ , because if only  $x$  questions are allocated to the subtree, then we cannot get a smaller  $wc_{\text{case}}$  than  $R$ . In any case, every sub-tree at depth  $l$  needs at least  $x + 1$  questions. Consider the case when no node at depth  $l$  is asked, instead, we ask nodes at depth  $l + 1$  or greater. Once again the  $wc_{\text{case}} \geq R$  (all answers as NO). Therefore, the proposed solution is the optimal one.

Finally, to see that  $y < m^l$  extra questions do not affect the worst case, notice that there is at least one sub-tree at depth  $l$  that will not be allocated an extra question — due to that sub-tree,  $wc_{\text{case}}$  will stay the same.

Algorithm 1 is the optimal algorithm with constant time complexity. It simply computes the maximum level that can be covered and then selects all the nodes in that level. Subsequently, it selects equal number of nodes in each of the sub-trees at depth  $l$  to ask questions.  $\square$

## A.9 Proof of Theorem 4.11

**PROOF.** We augment the upward-forest with a single root node such that there exists an edge to the new root node from the root of each of the trees in the directed forest. Let the original forest be  $F$  and the new augmented tree (a upward-tree) be  $T$ . Let  $N_T$  be an optimal selection of  $k$  nodes from  $T$  at which questions are asked. We first convert  $N_T$  into  $N'_T$  such that there is no question asked at the root. (We delete the root from  $N_T$ , if present.) Since a question at the root affects the worst-case candidate set at  $T$ , denoted by  $wc_{\text{case}_T}$ , at most by 1, we have  $wc_{\text{case}_T}(N'_T) \leq wc_{\text{case}_T}(N_T) + 1$ .

Let  $N_F$  be the optimal solution on  $F$ . If we select set  $N_F$  as the questions to be asked at  $T$ , we get  $wc_{\text{case}_T}(N_T) \leq wc_{\text{case}_T}(N_F)$ , since  $N_T$  gives the optimal worst-case candidate set for  $T$ . Also, since there is an addition of at most one node to the worst-case candidate set when  $N_F$  is used on  $T$  instead of  $F$ ,  $wc_{\text{case}_T}(N_F) \leq wc_{\text{case}_F}(N_F) + 1$ , where  $wc_{\text{case}_F}$  is the worst-case candidate set when the questions are asked at the nodes in  $F$ . Now, apply  $N'_T$  on  $F$ . We then have  $wc_{\text{case}_F}(N'_T) \leq wc_{\text{case}_T}(N'_T) \leq wc_{\text{case}_T}(N_T) + 1 \leq wc_{\text{case}_F}(N_F) + 2$ .

Thus, choosing the questions to minimize worst-case candidate set in the tree gives the optimal worst-case candidate set for the forest plus at most two more nodes. We ignore the additive factor of 2 in our calculations.  $\square$

## A.10 Upward-Forests

### A.10.1 Overview

First, we define the subtree of  $x$  to be  $x$  along with all nodes below  $x$ , i.e., all  $x'$  such that  $x \in rset(x')$ .

We use a dynamic programming algorithm<sup>4</sup>, listed in Algorithm 2, but the steps are explained next. The algorithm builds an array  $T_x[i]$ , for each node  $x \in V$ , and  $i \in \{0, \dots, k\}$ , bottom-up. Intuitively, in  $T_x[i]$ , we maintain a set of options for asking questions at the children  $y$  and  $z$  of  $x$ , by recording the worst-case contributions to the candidate set for various locations of the target node for each option, as described below:

Formally, we define array  $T_x[i]$  to contain a set of 4-tuples of the following form  $(K, P, N, R)$ , where  $P$  and  $K$  are pairs of values while  $N, R$  are singleton values. Consider a 4-tuple  $((k_1, k_2), (p_1, p_2), n, r)$ : This 4-tuple indicates that when  $i$  questions are allocated to node  $x$ , there is a configuration of asking questions in  $x$ 's subtree, along with  $k_1$  questions in  $y$ 's subtree, and  $k_2$  questions in  $z$ 's subtree, such that no matter where the target node is, the worst possible contribution to the overall worst-case candidate set coming

<sup>4</sup>While the algorithm is only described for 2-ary trees, it can be easily generalized to handle  $m$ -ary trees.

---

### Algorithm 2: Single-Bounded Upward-Forest

---

**Data:**  $\mathcal{G}$  =upward-tree,  $k$  = budget

**Result:** optimal set of  $k$  nodes to ask questions to

**for all nodes  $x$  in  $\mathcal{G}$ , bottom-up do**

```

 $T_x := \emptyset;$ 
 $T_x[0] := \{((0, 0), (size(x), 0), 0, size(x))\};$ 
if  $x$  has 1 or 2 children then
   $y :=$  left sub-child of  $x;$ 
   $z :=$  right sub-child of  $x;$ 
  for  $i : 0 \dots k$  do
    for all  $k_1, k_2 : k_1 + k_2 = i$  do
      for all  $((*, *), (p_1, p_2), n, r) \in T_y[k_1]$  and all  $((*, *), (p'_1, p'_2), n', r') \in T_z[k_2]$  do
         $p_b := \max\{p_1, p_2 + n', p'_1, p'_2 + n\};$ 
         $n_a := n + n';$ 
         $r_a := p_a := 1;$ 
        if  $k_1 == 0 \wedge k_2 \neq 0$  then
           $p_a := \max\{p'_1 + 1, r' + 1\};$ 
           $p_b := \max\{n', p'_2\} + size(y);$ 
           $r_a := r' + 1;$ 
        if  $k_2 == 0 \wedge k_1 \neq 0$  then
           $p_a := \max\{p_1 + 1, r + 1\};$ 
           $p_b := \max\{n, p'_2\} + size(z);$ 
           $r_a := r + 1;$ 
        if  $x$  has one child  $y$  then
           $T_x[i + 1] := T_x[i + 1] \cup \{((i, 0), (1, \max\{p_1, p_2\}), n_a, 1)\};$ 
          if  $i \neq 0$  then
             $T_x[i] := T_x[i] \cup \{((i, 0), (p_a, p_2), n_a, r_a)\};$ 
        else
           $T_x[i + 1] := T_x[i + 1] \cup \{((k_1, k_2), (1, p_b), n_a, 1)\};$ 
          if  $i \neq 0$  then
             $T_x[i] := T_x[i] \cup \{((k_1, k_2), (p_a, p_b), n_a, r_a)\};$ 
        compress  $T_x[i];$ 
  else
    for  $i : 1 \dots k$  do
       $T_x[i] := \{((k, 0), (1, 0), 0, 1)\};$ 

```

$r :=$  root of tree;

$t :=$  tuple in  $T_r[k]$  that has smallest  $p_1$ ;

trace the origin of  $t$  until the leaves of the tree;

output the questions;

---

from  $x$ 's subtree corresponds to one of  $p_1, p_2, n, r$ . That is, if we were to ask questions at a node set  $N$  in  $x$ 's subtree, the worst contribution from  $x$ 's subtree to the overall candidate set would be one of these numbers. Value  $p_1$  corresponds to the worst contribution such that (a) it contains the root  $x$  and (b) the target node is present in  $x$ 's subtree,  $p_2$  corresponds to the worst contribution such that (a) it does not contain  $x$  and (b) the target node is present in  $x$ 's subtree,  $n$  corresponds to the contribution when the target node is neither present in  $x$ 's subtree nor is in  $rset(x)$ . (There is only one number as against two for  $P$ , since the contribution to the candidate set will never contain the root.) while  $r$  corresponds to the contribution when the target node is not present in  $x$ 's subtree but is reachable from  $x$ . (There is only one number since the contribution to the candidate set will always contain the root.)

The array  $T_x[i]$  is thus a collection of worst-case tuples. In the worst case, we might need to maintain upto  $O(n^4)$  tuples (corresponding to all combinations of the last 3 entries in the tuple). However, we can also use the following rule to discard tuples: If

all values in the last three entries in the first tuple are greater than or equal to the corresponding values in the second tuple, we can discard the first tuple in favor of the second because the second allocation is better overall than the first. This approach corresponds to maintaining the minimum-skyline of these tuples. However, this procedure is not necessary for correctness.

### A.10.2 Details

We now consider the details of the algorithm for upward-forests. Let the node  $x$  have two children  $y$  and  $z$ . We now describe how to create  $T_x[i]$  using  $T_y[*]$  and  $T_z[*]$ . For the trivial case when  $i = 0$ , we define:  $T_x[0] = ((0, 0), (size(x), 0), 0, size(x))$ , where  $size(x)$  gives the size of  $x$ 's subtree. Intuitively, all values are  $size(x)$  since no node can be eliminated from the tree, and therefore the contribution to the candidate set is of  $size(x)$ , except when the root is specifically excluded, i.e., for  $p_2$  and  $n$ .

Now consider  $i > 1$ . Let  $k_1$  be the number of questions asked at  $y$ 's subtree and  $k_2$  be the number asked at  $z$ 's subtree. If  $i - k_1 - k_2 = 1$ , then a question is asked at  $x$ , otherwise  $i = k_1 + k_2$ .

For every possible splitting of  $i$  into  $k_1$  and  $k_2$ , we add new tuples to  $T_x[i]$  using  $T_y[k_1]$  and  $T_z[k_2]$ . The first entry of the tuple will be  $(k_1, k_2)$ . Consider the 3-tuple  $((p_1, p_2), n, r)$  in  $T_y[k_1]$  (corresponding to the last 3 entries in the tuple) and in  $T_z[k_2]$ :  $((p'_1, p'_2), n', r')$ , creating a new 3-tuple for  $T_x[i]$ :  $((p_a, p_b), n_a, r_a)$ .

**Case 1: none at root, both  $k_1, k_2$  nonzero:** In this case, some questions have been asked at both  $y$  and  $z$ , i.e.,  $k_1$  and  $k_2 \neq 0$ .

First, let us consider  $p_a$ : In this case, the target node is either in  $y$ 's subtree or  $z$ 's subtree or is  $x$  itself. Then, in order for the worst-case contribution to contain the root  $x$ , all the answers at both  $y$  or  $z$ 's subtrees need to be YES. Then, the only case is when the target node is  $x$  (because in all other cases, at least one of the answers would be NO). Here, the worst-case contribution is simply 1.

For  $p_b$ , since some questions were asked at both  $y$  and  $z$ , at least one of these questions has to have been NO, eliminating the path to the root. Thus, the worst-case contribution is either when the target node is present in  $y$ 's subtree, giving  $\max(p_1, p_2 + n')$ , or when the target node is present in  $z$ 's subtree, giving  $\max(p'_1, p'_2 + n)$ .

For  $n_a$ , none of the answers were YES, as a result, the worst-case contribution does not contain either  $y$  or  $z$  or  $x$ , and is simply the combination of contributions  $n$  and  $n'$ . For  $r_a$ , all of the answers will be YES, as a result of which the worst-case contribution always contains simply the root node  $x$ . (Thus  $r_a = 1$ .)

**Case 2: none at root, one of  $k_1, k_2$  zero:** Now consider the case when one of the  $k_i$  is 0. Let  $k_1 = 0$ , while  $k_2 \neq 0$ . In this case, unless we get a YES answer from one of the nodes in  $z$ 's subtree, all of  $y$ 's sub-tree will form part of the candidate set. Consider  $p_a$ . If the target node is present in  $y$ 's subtree then any question in  $z$ 's subtree will give NO, eliminating  $x$ . Therefore, this situation does not contribute to  $p_a$ . If the target node is in  $z$ 's subtree, then the worst case is  $p'_1 + 1$  (neither  $z$  nor  $x$  is eliminated, and since all the answers are YES,  $y$ 's subtree is removed). If the target node is  $x$ , then the worst-case contribution is  $r' + 1$ .

Consider  $p_b$ . In this case, we receive at least one NO answer in order to eliminate the path to the root  $x$ , this NO answer can only come from  $z$ 's sub-tree. So we have two cases. Either the target node is in  $y$ 's subtree or is in  $z$ 's subtree. In the first case, we have  $size(y) + n'$ , and in the second case, we have:  $p'_2 + size(y)$ .

For  $n_a$ , we cannot eliminate the  $y$ 's sub-tree, therefore, we have  $n' + size(y)$ . For  $r_a$ , the answers will be YES for  $z$ 's subtree, therefore, we eliminate  $y$ 's subtree, and we get  $r' + 1$ .

**Case 3: root, both  $k_1, k_2$  nonzero:** Consider the case when we ask the root. Let  $k_1$  and  $k_2 \neq 0$ . First, let us consider  $p_a$ : In this case, the target node is either in  $y$ 's or  $z$ 's subtree or is  $x$  itself. Then, in

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### Algorithm 3: Single-Bounded Balanced Upward-Tree

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**Data:**  $\mathcal{G} = m$ -ary balanced upward-tree with depth  $d$ ,  $k$   
**Result:** optimal set of  $k$  nodes to ask questions to  
 $\alpha := \lceil \log_m k \rceil$ ;  
**if**  $k \neq m^\alpha$  **then**  
     $\alpha := \alpha + 1$ ;  
 $N := \emptyset$ ;  
**while**  $|N| < k$  **do**  
    pick a new node  $n$  at depth  $\alpha$ ;  
    pick a leaf  $l$  in  $n$ 's subtree;  
     $N := N \cup \{l\}$ ;  
**return**  $N$ ;

---

order for the worst case to contain the root, all the answers at both  $y$  or  $z$  need to be true. Then, the only case is when the target is  $x$  (because in all other cases, at least one of the answers would be NO). Here, the worst-case contribution is 1.

In the case of  $p_b, n_a$  and  $r_a$ , the equations stay the same as above, since for  $p_b$  and  $n_a$  asking the root does nothing, while for  $r_a$  the contribution is already simply the root.

**Case 4: root, one of  $k_1, k_2$  zero:** Now let  $k_1 = 0$ , while  $k_2 \neq 0$ . Since we ask a question at the root, the only way  $p_a$  is affected is if the root is the target. In this case,  $p_a = 1$ . Consider  $p_b$ . In this case, we can either have the target node in  $y$ 's subtree or  $z$ 's subtree. In the first case, we have  $size(y) + n'$ , and in the second case, we have:  $p'_2 + size(y)$  or  $p'_1$ . For  $n_a$ , we cannot eliminate  $y$ 's subtree, therefore, we have  $n' + size(y)$ . For  $r_a$ , we get 1.

**Case 5: root, both  $k_1, k_2$  zero:** If we have both  $k_1 = k_2 = 0$ , then we have  $p_a = r_a = 1, p_b = n_a = size(y) + size(z)$ .

The cases when there is only a single child of the node  $x$  is similar, and we do not discuss it here. If there is only a single child  $y$  of the node  $x$ , then we create a virtual child  $z$  of  $x$  which has a single tuple  $T_z[0] = ((0, 0), (0, 0), 0, 0)$ . (This allows us to simplify the pseudo-code in Algorithm 2.) Note that the **compress** procedure in the pseudo-code effectively maintains the minimum-skyline of the tuples in  $T_x[i]$ .

**PROOF OF THEOREM 4.12:** Algorithm 2 does a bottom up computation of 4-tuples per node as described above. To see that the complexity is indeed  $O(k^2 n^5)$ , we first prove that Algorithm 2 maintains for each node, for each  $(k_1, k_2)$  pair,  $O(n^2)$  tuples.

We use an inductive argument. The claim is clearly true for the base case, i.e., the leaves. Now, consider an internal node  $x$ . For a given  $(k_1, k_2)$  split, if it falls into case 1, 3, or 5, two of the items among  $\{p_a, p_b, n_a, r_a\}$  are 1, and therefore there can be at most  $O(n^2)$  tuples. For case 2 or 4, we merely copy the tuples from one of the children of the node  $x$  (by modifying the tuple), therefore, we once again have  $O(n^2)$  tuples. The bottom up procedure takes  $O(n^4)$  time to compute the tuples corresponding to  $(k_1, k_2)$  for a parent node given its children, and the theorem follows.  $\square$

If the arity is  $> 2$ , then we generate  $T_x$  by combining the tuples iteratively over the children of  $x$  (i.e., the first  $i-1$  children with the  $i$ th child of  $x$  — allocating  $k_1$  questions to the first  $i-1$  children and  $k_2$  questions to subtree of the  $i$ th child of  $x$ ), thus the worst case complexity is  $O(m \cdot k^2 \cdot n^5)$ , where  $m$  is the arity. The rules for combining tuples are similar to the ones for arity 2.

## A.11 Balanced Upward-Trees

We now present an efficient algorithm for an balanced upward-tree. The following theorem formalizes our results.

**THEOREM A.2 (BALANCED UPWARD-TREE).** *For an balanced upward  $m$ -ary tree of large depth  $d$ , Algorithm 3 finds the optimal solution for the Single-Bounded problem in  $O(1)$ .*

**PROOF.** We now provide the proof of optimality of Algorithm 3.

Let the optimal solution be different from that suggested by Algorithm 3. Since we are given that  $d$  is large relative to  $k$ , the case when the candidate set is the largest for this solution is the one when all the nodes return NO. (If any of the nodes returns YES, then the target node is on the path from the node to the root, which is of length at most  $d + 1$ . When all the nodes return NO, the size of the candidate set is at least  $(m^{d+1} - 1)/(m - 1) - k \times (d + 1)$ , which is much larger than  $d + 1$ .) Note that the optimal solution will only contain leaves, because leaves let us eliminate the most nodes from the candidate set on getting a NO answer.

Therefore, in the optimal solution, the set of nodes at which questions are asked are the ones such that the union of their paths to the root has the largest size. Note that this largest size is upper bounded by  $k \times (d + 1)$ , since each path can be of length at most  $d + 1$ . Also note that if we picked leaf nodes, we can get very close to  $k \times (d + 1)$ : Consider the nodes at the depth  $\alpha$  such that  $m^\alpha < k \leq m^{\alpha+1}$ . Pick one leaf from the subtree corresponding to each node at depth  $\alpha + 1$  (until  $k$  leaves are picked). The size of the union in this case is  $k \times (d - \alpha) + (m^{\alpha+1} - 1)/(m - 1) = k \times (d + 1) - (k(\alpha + 1) - (m^{\alpha+1} - 1)/(m - 1))$ . We first prove that in any optimal solution, any two paths from the nodes picked to the root in this solution meet at depth  $\leq \alpha$ .

Consider an optimal solution where there exist two leaves  $a$  and  $b$  such that their paths to the root meet at a depth  $x > \alpha$ . In this case, there exists a node  $c$  at depth  $\alpha + 1$  that has no leaves at which questions are being asked. By transferring the question from the leaf  $a$  to a leaf under  $c$ , we can only reduce the size of the worst-case candidate set. Thus this solution is not optimal.

Notice also that all solutions of size  $k$  where the paths from the leaves meet at depth  $\leq \alpha$  have the same size of union. Thus, this solution is optimal.  $\square$

## B. Single-Unlimited

### B.1 Proof of Theorem 4.15

PROOF. The proof involves repeating runs of Single-Bounded to find the smallest  $N$  such that the worst-case candidate set is of size 1. Below we show two approaches, that are combined by running one step of each until one of them terminates.

The first approach involves binary search. Here, we let  $|N|$  vary between 1 and  $n$ , run Single-Bounded and stop once the worst case candidate set size is 1 for  $|N| = k$ , but not when  $|N| = k - 1$ . This process takes  $O(\log n)$  time.

The second approach involves doubling  $|N|$  each time starting from 1, i.e.,  $|N| = 1, 2, 4, 8, \dots$  and running Single-Bounded every time. Once we find a  $k$  such that Single-Bounded when  $|N| = k$  has worst case candidate set 1, then we repeat the procedure of doubling from  $|N| = k/2$ , i.e.,  $|N| = k/2 + 1, k/2 + 2, k/2 + 4, \dots$ , recursively, until we find two consecutive  $|N|$  such that one of them has worst case candidate set of 1 and the other one does not. Thus, we will do  $\log o$  steps every time, and we reduce the space of search by 1/2 every time, leading to  $\log o$  such steps. Thus, we have a complexity of  $\log^2 o$ .  $\square$

### B.2 Proof of Theorem 4.16

PROOF. Consider all leaf nodes across all trees in the forest. Note that each leaf node has a single parent. Firstly, it is easy to see that we need to ask all leaves. If not, consider a leaf node  $a$ . Let its parent be  $b$ . If we do not ask  $a$ , but  $b$  returns YES, (but no other child of  $b$  returns YES) then it is not clear if the target node is  $a$  or  $b$ . Therefore, we need to ask  $a$ . The same argument can be used for all leaves. Now assume all questions at leaves return a NO answer (effectively, leaves can be removed from the tree.)

---

### Algorithm 4: Single-Unlimited Upward-Tree

---

**Data:**  $\mathcal{G} = m$ -ary upward-tree,  $k$   
**Result:** optimal set of  $k$  nodes to ask questions to  
**if**  $\exists$  a leaf  $f$  such that  $f$ 's parent has degree  $\neq 2$  **then**  
     $N =$  all leaves except  $f$ ;  
**for** all internal nodes  $x$  **do**  
    **if**  $x$  has indegree 1 **then**  
         $N = N \cup \{x\}$ ;  
**return**  $N$ ;

---

The argument can be repeated for each parent of the leaves as well. We then arrive at the roots of each of the trees in the forest and their children. We cannot leave two of the roots unasked because we cannot distinguish between them without asking a question at both of them. However, we can avoid asking a question at one of the roots because if we get a NO response from all other trees as well as children of that node, then that node has to be the target node.  $\square$

## B.3 Proof of Theorem 4.17

PROOF. (Refer Algorithm 4 for the pseudo-code.) We apply the standard conversion of an upward-forest into an upward-tree. First, consider the case when two or more leaves (out of a total of  $r$ ) are left unasked. Then either of them could be the target node. (To see this, assign NO answers to all other nodes in the tree. Then either of the two unasked leaves could return a YES answer.) Therefore at least  $r - 1$  leaves need to be asked. Note that if every parent of a leaf has indegree = 2, then all  $r$  leaves have to be asked. If not, then the one leaf that is excluded is one whose parent does not have indegree = 2. We will prove later that the answers on asking questions at the remaining leaves will help us infer the answer to the question at the unasked leaf. For now, assume that we know the answers to questions at every leaf.

We now prove that we definitely need to ask any node whose indegree is 1. Consider any path  $a_1 \rightarrow a_2$  where  $a_1$  has an indegree of  $\geq 1$ , and  $a_2$  has an indegree of 1. In this scenario, even if we know the answer of  $a_1$ , we need to ask  $a_2$  because if  $a_1$  returned YES, there is no way we can distinguish between  $a_1$  and  $a_2$ . Thus, we need to ask  $a_2$ .

Notice that if the answers to questions at all other nodes are known, then it is not necessary to ask any node whose indegree is greater than 1, since its answer can be inferred from its children, bottom up. If all the children of a node return YES, then the node must return YES, and NO otherwise. This coupled with the previous statement shows that all and precisely those internal nodes that have indegree 1 need to be asked questions.

Now, we return to the point of inferring the answer at the leaf where a question is not asked. We do so in the following manner. If that leaf is connected to a parent whose indegree is 1, then the parent itself will be asked. If all answers are NO, including the parent, then the unasked leaf is the target node. Else, the unasked leaf will return precisely what the parent returns.

If the leaf is connected to a parent whose indegree  $> 2$ , then if all other leaves connected to the parent return YES, then the given leaf returns YES (since there is a single target node). If all nodes everywhere return NO, then the unasked leaf is the target node. In all other cases, the leaf returns NO. (Note that if all parents of leaves had indegree 2, then for the unasked leaf, even if its sibling returns YES, it may be the case that either the sibling or the parent is the target node, since the parent is never asked in this case.)  $\square$

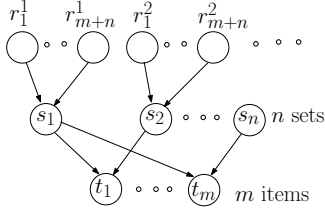


Figure 6: Hardness Proof for Multi-Bounded

## C. Multi-Bounded

### C.1 Proof of Theorem 5.2

PROOF. We give a reduction from the NP-hard max-cover problem: Given  $m$  items,  $n$  sets, and an integer  $k$ , the goal is to choose  $k$  sets that cover the most number of items.

Given an instance of the max-cover problem, we construct an instance of Multi-Bounded with the following DAG containing three layers of nodes, as depicted in Fig. 6. The second and third layers of nodes are identical to the first and second layers in the proof for Theorem 4.4: The second layer has one node corresponding to each set and the third layer has a node corresponding to each item. There is an edge from node  $s_i$  to the node  $t_j$  iff item  $t_j \in s_i$  in the max-cover instance. For each node  $s_i$  in the second layer, we add  $m+n$  unique parents in the first layer,  $r_1^i, r_2^i, \dots, r_{m+n}^i$ , with an edge to  $s_i$ . We want to solve Multi-Bounded on this constructed DAG for  $k$  questions. Note that to solve the Multi-Bounded problem, we will always pick nodes corresponding to sets, because they let us eliminate the maximum number of nodes corresponding to items, in the worst case. If we pick a node from the first or the third layer, we will get a YES or NO response respectively in the worst case, allowing us to eliminate only one node from the candidate set. On the other hand, a node corresponding to a set allows us to eliminate either  $m+n$  nodes (on YES) or the number of nodes corresponding to items covered by that set (on NO). Thus, we only pick nodes corresponding to sets. In the worst case, none of the answers returned are YES. If any of them were, then we would eliminate  $m+n$  nodes per YES answer from the candidate set. On the other hand, if those answers were NO, we would eliminate less than  $m$  nodes corresponding to items. Thus, the worst-case candidate set occurs when all answers are YES. In order to improve the worst case, we would like to cover as many nodes from the  $m$  nodes corresponding to items by picking nodes corresponding to sets. Thus the nodes corresponding to sets that are picked in Multi-Bounded precisely correspond to the sets that are picked in the max cover problem.  $\square$

### C.2 Proof of Theorem 5.3

PROOF. Given an instance of the decision version of the Multi-Bounded problem, we can express it as an instance of  $\Sigma_2^P$  in the following way:

$$\exists y_1 \forall y_2 [L(y_2) \vee (R(y_1, y_2) < X)],$$

where  $y_1$  corresponds to a set of nodes at which questions are asked,  $y_2$  corresponds to all possible instances of the target set.  $L(y_2)$  checks in PTIME whether  $y_2$  contains two nodes with a path from one to the other: If so, it returns YES.  $R(y_1, y_2)$  evaluates the candidate set given  $y_1$  and  $y_2$ .  $\square$

### C.3 Proof of Theorem 5.4

PROOF. Consider the answers to a set of questions asked at nodes in a downward-tree. If we were to complement each answer (YES  $\rightarrow$  NO and NO  $\rightarrow$  YES), we would obtain the answers to the questions asked at nodes in the same tree if the direction of each edge was reversed (i.e., the upward-tree) for the same target set.

---

### Algorithm 5: Multi-Bounded Downward/Upward-Forest

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**Data:**  $\mathcal{G}$  =upward-tree,  $k$  = budget

**Result:** optimal set of  $k$  nodes to ask questions to

**for all nodes  $x$  in  $\mathcal{G}$ , bottom-up do**

```

 $T_x := \emptyset;$ 
 $T_x[0] := \{(0, 0), size(x), 0, size(x)\};$ 
if  $x$  has 1 or 2 children then
   $y :=$  left sub-child of  $x;$ 
   $z :=$  right sub-child of  $x;$ 
  for  $i : 0 \dots k$  do
    for all  $k_1, k_2 : k_1 + k_2 = i$  do
      for all  $((*, *) , p_1, p_2, n)$  in  $T_y[k_1]$  and all  $((*, *) , p'_1, p'_2, n')$  in  $T_z[k_2]$  do
         $p_a :=$ 
           $\max\{p_1 + n', p'_1 + n, p_1 + p'_1, n + n'\} + 1;$ 
         $p_b := \max\{p_2 + p'_2, p_2 + p'_1, p'_2 + p_1, p_2 + n', p'_2 + n\};$ 
         $n_a := n + n' + 1;$ 
        if  $k_1 = 0 \wedge k_2 \neq 0$  then
           $p_b := p'_2 + n;$ 
        if  $k_2 = 0 \wedge k_1 \neq 0$  then
           $p_b := p_2 + n';$ 
         $T_x[i+1] :=$ 
           $T_x[i+1] \cup \{((k_1, k_2), p_a, p_b, 0)\};$ 
        if  $i \neq 0$  then
           $T_x[i] :=$ 
             $T_x[i] \cup \{((k_1, k_2), p_a, p_b, n_a)\};$ 
      compress  $T_x[i];$ 

```

**else**

```

for  $i : 1 \dots k$  do
   $T_x[i] := \{(k, 0), 1, 0, 0\};$ 

```

$r :=$  root of tree;

$t :=$  tuple in  $T_r[k]$  that has smallest  $p_1$ ;

trace the origin of  $t$  until the leaves of the tree;

output the questions;

---

(Recall from Theorem 3.3 that each question either rules out the ancestors or the descendants from the candidate set.) Thus, we can solve Multi-Bounded on a downward-tree by reversing all edges and solving it on the upward-tree. The converse is also true. Hence the problems are equivalent.  $\square$

### C.4 Downward/Upward-Forests:

We now describe the algorithm for solving Multi-Bounded on Downward/Upward forests in detail. We describe the algorithm for a tree with arity 2, however our approach generalizes to trees with different arities.

We set  $T_x[0] = ((0, 0), size(x), 0, size(x))$ , since the worst-case contribution to the candidate set when no questions are asked is always the entire tree under  $x$ , except 0 in the case of  $p_2$ , since the root can never be eliminated when no questions are asked. We define the subtree under  $x$  to be  $rset(x)$ .

Now, we generate rules to generate a triple  $(p_a, p_b, n_a)$  by combining triples from  $T_y[k_1] : (p_1, p_2, n)$  and  $T_z[k_2] : (p'_1, p'_2, n')$ .

**Case 1: None at root,  $k_1, k_2$  nonzero.** In this situation, for  $p_a$ , none of the answers in any of the sub-trees can be YES. If any answer in either sub-tree is YES, then the root node  $x$  is automatically excluded. Therefore, none of the answers is YES, and target nodes can be in the subtree under  $y$  or  $z$  or both or be  $x$  itself.

If there are target nodes under  $y$  but none under  $z$ , and if none of the answers are YES, we have a contribution of  $p_1$  towards the candidate set from the sub-tree under  $y$ . Additionally, the sub-tree under  $z$  will make a contribution of  $n'$  towards the candidate set

(since there are no target nodes under  $z$ ). Therefore, we have a overall contribution of  $p_1 + n' + 1$ . Similarly, if there are target nodes under  $z$  but not  $y$ , then the contribution is  $p'_1 + n + 1$ . If the target node is  $x$  itself, then we get a contribution to candidate set of size  $n + n' + 1$ , since the target nodes are not present in the sub-trees under  $y$  or  $z$ .

If there are target nodes under both  $y$  and  $z$ , then we have a contribution of  $p_1 + p'_1 + 1$ . Therefore, in total, we have  $p_a = \max\{p_1 + n' + 1, p'_1 + n + 1, p_1 + p'_1 + 1, n + n' + 1\}$ .

For  $p_b$ , at least one of the answers has to be a YES, only then will the root  $x$  be eliminated. Thus, the cases that we have are either that the YES came from the sub-tree under  $y$ , or the YES came from the sub-tree under  $z$  or both. Therefore, at least one component of the contribution to the candidate set must be either  $p_2$  or  $p'_2$ . Thus, the contribution could be:  $p_2 + p'_2$  (when there are YES answers in both sub-trees),  $p_2 + p'_1, p'_2 + p_1, p_2 + n', p'_2 + n$  (when there are YES answers from only one sub-tree). For  $n_a$ , we simply have worst-case contribution to be  $n + n' + 1$ .

**Case 2: None at root,  $k_1$  zero,  $k_2$  nonzero.** For  $p_a$ , the root will remain iff there is no YES answer to a question in the sub-tree under  $z$ . Since we have  $p_1 = n = \text{size}(y)$ , the size of the sub-tree under  $y$ , the same equation applies, i.e., the sub-tree under  $y$  contributes  $p_1$  or  $n$ , while the sub-tree under  $z$  contributes  $p'_1$  or  $n'$ , and  $x$  contributes 1.

For  $p_b$ , at least one of the answers has to be a YES, only then will the root be eliminated. Thus, a YES has to come from the sub-tree under  $z$ . Therefore,  $z$  has to contribute  $p'_2$ , while  $y$  contributes  $\text{size}(y)$  to the candidate set. For  $n_a$ , we have worst-case contribution to be  $n + n' + 1$ .

**Case 3: root,  $k_1, k_2$  nonzero.** For  $p_a$ , the root will remain in the candidate set iff there is no YES answer to a question under either of the sub-trees. Therefore, the target node can either be the root, or under  $y$  or  $z$  or both. Thus, we have, as before,  $p_a = \max\{p_1 + n' + 1, p'_1 + n + 1, p_1 + p'_1 + 1, n + n' + 1\}$ .

For  $p_b$ , the answer from the root has to be a YES. Additionally, since the root is eliminated, at least one other answer is a YES. Thus, we have the same potential worst-case contributions:  $p_2 + p'_2$  (when there are YES answers in both sub-trees),  $p_2 + p'_1, p'_2 + p_1, p_2 + n', p'_2 + n$  (when there are YES answers from only one sub-tree). For  $n_a$ , since we get a NO answer from the root,  $n_a = 0$ .

**Case 4: root,  $k_1$  zero,  $k_2$  nonzero.** For  $p_a$ , the root will remain in the candidate set iff there is no YES answer to a question under either of the sub-trees. Therefore, the target node can either be the root, or under  $y$  or  $z$  or both. Thus, we have, as before,  $p_a = \max\{p_1 + n' + 1, p'_1 + n + 1, p_1 + p'_1 + 1, n + n' + 1\}$ . (The same equation works since  $p_1 = n = \text{size}(y)$ .)

For  $p_b$ , the answer from the root has to be a YES. Additionally, since the root is eliminated, at least one other answer is a YES. Thus, we have a contribution of  $p'_2$  from  $z$  and a contribution of  $\text{size}(y)$  from  $y$ . For  $n_a$ , since we get a NO answer from the root,  $n_a = 0$ .

**Case 5: root,  $k_1, k_2$  zero** For  $p_a$ , the root will remain in the candidate set. However, the same equation holds, i.e.,  $p_a = \max\{p_1 + n' + 1, p'_1 + n + 1, p_1 + p'_1 + 1, n + n' + 1\}$ , since  $p_1 = n = \text{size}(y)$  and  $p'_1 = n' = \text{size}(z)$ .

For  $p_b$ , the answer from the root has to be a YES. Since there is no way the root is getting eliminated, we get  $p_b = 0$ . For  $n_a$ , since we get a NO from the root,  $n_a = 0$ .

For the case when  $x$  has a single child  $y$ , if we create a dummy node  $z$  with a single tuple  $T_z[0] = ((0, 0), 0, 0, 0)$ , then the same equations given above hold for each of the cases.

The approach above can be generalized to  $m$ -ary trees in a manner similar to that for Single-Bounded Upward-Forests.

## C.5 Balanced Trees

We now consider solving Multi-Bounded for the special case of balanced trees. (Recall we don't need to consider upward and downward-trees separately.) The following theorem shows that for small  $k$  (compared to the number of nodes in the tree), there is a constant-time solution for a balanced  $m$ -ary tree.

**THEOREM C.1 (BALANCED TREES).** *Multi-Bounded on a balanced  $m$ -ary tree of height  $d$  is solvable in  $O(1)$  when  $k \leq m^{\frac{d}{2}}$ .*

Informally, the approach is to pick nodes at a depth such that the path to the root is equal to the size of the sub-tree underneath (such that a YES answer or a NO answer would eliminate roughly the same number of nodes from candidate set), in order to minimize wcase.

**PROOF.** We solve Multi-Bounded for a balanced downward-tree; the case of the balanced upward-tree is identical (using Theorem 5.4).

We pick nodes such that the size of the candidate sets when all answers are YES or when all answers are NO need to be made equal, i.e., pick nodes at a certain depth so that the number of descendants of a node is equal to the number of ancestors. The number of descendants for a node at depth  $\alpha$  is  $(m^{d-\alpha+1} - 1)/(m - 1)$  and the number of ancestors is  $\alpha$ . Therefore, we ask questions at nodes at level  $\alpha$  such that  $\alpha = (m^{d-\alpha+1} - 1)/(m - 1)$ . Clearly,  $\alpha$  is large enough to have at least  $k$  nodes at the level: If  $\alpha$  were small, then the size of the sub-tree would be far larger than the path from the root. (As an aside, note that the  $k$  paths to the root from the chosen nodes share  $\leq \log_m k$ , but that is negligible and doesn't affect our solution.)

The proof of the optimality of our solution follows using a simple argument: For any other solution of  $k$  nodes  $N_k$ , consider the wcase( $N_k$ ) obtained based on answering "YES" to all question at nodes at or above level  $\alpha$ , and "NO" to all other questions. It can be seen easily, that for each node in  $N_k$ , its contribution to the size of the candidate set is at least as large as  $\alpha$ , and thus our solution is no worse than  $N_k$ .

Note also that if we ask any questions at a depth  $< \alpha$ , the candidate set size goes up at least  $m$  times (on returning a YES for that question), therefore our solution is optimal.  $\square$

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### Algorithm 6: Multi-Bounded Balanced Trees

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**Data:**  $\mathcal{G} = m$ -ary balanced tree with depth  $d, k$   
**Result:** optimal set of  $k$  nodes to ask questions to  
 $l := \lfloor \log_m k \rfloor$ ;  
 $N := \emptyset$ ;  
**while**  $|N| < k$  **do**  
    pick a new node  $n$  at depth  $l$ ;  
    pick a node  $x$  at level  $\alpha$  in subtree under  $n$ ;  
     $N := N \cup \{x\}$ ;  
**return**  $N$ ;

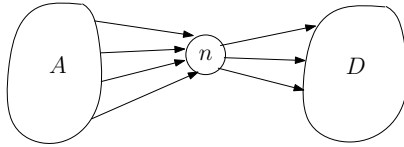
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## C.6 Balanced Forest

We can generalize the results for balanced trees to obtain a better solution for *balanced forests*, i.e., a collection of balanced trees. The generalization to forests employs a dynamic programming algorithm operating over the balanced trees constituting the forest.

**THEOREM C.2 (BALANCED FOREST).** *Multi-Bounded for a balanced forest can be solved in  $O(k^2 r)$  where  $r$  is the number of trees.*

The algorithm iteratively (for each  $i$ ) computes the best allocation of questions to each of the balanced trees, by using the optimal allocation of  $k_1$  questions to the first  $i$  balanced trees, and  $k_2$  questions



**Figure 7: Triviality of Multi-Unlimited**

to the  $(i + 1)$ th tree, and combining the worst-case candidate sets (The candidate sets for each balanced tree are independent and can be combined.) to give one candidate allocation of  $k_1 + k_2$  questions to the first  $i + 1$  trees. We compute this value for all possible  $k_1$  and  $k_2$  such that  $k_1 + k_2 \leq k$ , and all  $i$ , recursively, and return the best possible allocation when  $i = r$ .

**PROOF.** We solve the balanced forests problem using dynamic programming. Given that we can compute the best set of up to  $k$  questions to ask the first  $i$  sub-trees, such that the worst-case candidate set is minimized, we consider the best set of up to  $k$  questions to ask the first  $i + 1$  trees.

Consider a total of  $j$  questions being asked at the first  $i + 1$  trees. Let  $j - a$  questions be asked at the first  $i$  sub-trees and  $a$  questions at the last tree. We can compute the best set of  $a$  questions to ask the last tree in  $O(1)$ . Additionally, the solution of the problem of finding the best  $j - a$  questions to ask the first  $i$  trees is already known to us. Thus, we can compute the best  $a$  such that the worst-case candidate set when  $j - a$  questions are asked at the first  $i$  trees and  $a$  questions are asked at the  $(i + 1)$ th sub-tree is the smallest.

Thus, we can compute the best set of up to  $k$  questions asked for the first  $i + 1$  trees for any  $i$  in  $O(k^2)$ , giving a total complexity of  $O(k^2 \cdot r)$  for the dynamic programming algorithm.  $\square$

## D. Multi-Unlimited

### D.1 Proof of Theorem 5.7

**PROOF.** Consider Fig. 7, abstractly representing a connected component of the input graph, focusing on any node  $n$ . We prove that we need to ask a question at  $n$  in order to ascertain if  $n \in U^*$ . Suppose we don't ask a question at  $n$ . Let the questions asked at all of the ancestors of  $n$ , i.e.,  $A$ , return YES, while questions at all of the descendants of  $n$ , i.e.,  $D$ , return NO. In this case, it is not clear if  $n$  is in  $U^*$  or not. It is possible that  $n$  is in  $U^*$ , in which case none of the nodes in  $A$  form part of  $U^*$ . Otherwise, if  $n \notin U^*$ , then there may be many nodes from  $A$  which are part of  $U^*$ .  $\square$