Inverting a multidimensional shape from moments

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1 Problem statement

The problem of reconstructing a function and/or its domain given its moments is encountered in many areas. Several applications from diverse areas such as probability and statistics [4], signal processing [10], computed tomography [8, 9], and inverse potential theory [1, 11] (magnetic and gravitational anomaly detection) can be cited, to name just a few. We can expound on some of these applications in a bit more detail. Consider the following diverse set of examples:

- A region of the plane can be regarded as the domain of a (uniform) probability density function. In this case, the problem is that of reconstructing, or approximating, the domain from measurements of its moments [4].
- Tomographic (line integral) measurements of a body of constant density can be converted into moments from which an approximation to its boundary can be extracted [9].
- Measurements of exterior gravitational field induced by a body of uniform mass can be turned into moment measurement, from which the shape of the region may be reconstructed [11].
- Measurements of exterior magnetic field induced by a body of uniform magnetization can yield measurement of the moments of the region from which the shape of the region may be determined [11].
- Measurements of thermal radiation made outside a uniformly hot region can yield moment information, which can subsequently be inverted to give the shape of the region [11].

In fact, inverse problems for uniform density regions related to general elliptical equations can all be cast as moment problems which fall within the scope of application of the results of this paper.

Although the reconstruction of a shape from its Radon transform is well-understood, the reconstruction of a shape from its moments is a problem that has only partially been solved. For instance, when the object is a polygon [5], or when it defines a quadrature domain in the complex plane [6], it has been proved that its shape can exactly be reconstructed from the knowledge of its moments. Both results deal with particular 2-dimensional shapes. For general n-dimensional shapes no inversion algorithm departing from the moments, is known. In order to explain the type of result we are looking for, we briefly repeat the inversion formula based on a shape's projections provided by the Radon transform.

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2 The Radon and Fourier integral transforms

The Radon transform $\mathcal{R}_{\vec{\lambda}}^{(f)}$ of a square-integrable n-variate function $f(\vec{x})$ with $\vec{x} = (x_1, \ldots, x_n)$ is defined as

$$\mathcal{R}_{\vec{\lambda}}^{(f)}(u) = \int_{\mathbb{R}^n} f(\vec{x})\, \delta(\vec{\lambda}\vec{x} - u)\, d\vec{x}$$

with $||\vec{\lambda}|| = 1$ and $\vec{\lambda}\vec{x} = u$ an $(n-1)$-dimensional manifold orthogonal to $\vec{\lambda}$. When $n = 2$, $\vec{\lambda}$ is fully determined by an angle θ and is given by

$$\mathcal{R}_{\theta}^{(f)}(u) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t,s)\, \delta(t\cos\theta + s\sin\theta - u)\, dt\, ds$$

For $n = 3$, $\vec{\lambda}$ is determined by angles θ and ϕ and

$$\mathcal{R}_{\theta,\phi}^{(f)}(u) = \int_{\mathbb{R}^3} f(t,s,v)\, \delta(t\cos\phi\cos\theta + s\cos\phi\sin\theta + v\sin\phi - u)\, dt\, ds\, dv$$

Making use of the celebrated Fourier slice theorem one obtains [7] that the one-dimensional Fourier transform of $\mathcal{R}_{\theta}^{(f)}(u)$,

$$\mathcal{F}_1^{\left(\mathcal{R}_{\theta}^{(f)}\right)}(z) = \int_{-\infty}^{+\infty} \mathcal{R}_{\theta}^{(f)}(u) \exp(-2\pi \mathtt{i} z u)\, du$$

equals the two-dimensional Fourier transform of the function f restricted to the straight line $(z\cos\theta, z\sin\theta)$:

$$\begin{aligned} \mathcal{F}_2^{(f)}(z\cos\theta, z\sin\theta) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t,s) \exp\left(-2\pi \mathtt{i} z(t\cos\theta + s\sin\theta)\right)\, dt\, ds \\ &= \mathcal{F}_1^{\left(\mathcal{R}_{\theta}^{(f)}\right)}(z) \end{aligned} \tag{1}$$

When $f(t,s)$ is the characteristic function of a compact set A in the complex plane, then (1) allows to reconstruct A, departing from its Radon transform, by taking the inverse two-dimensional Fourier transform of the right hand side of (1). In higher dimensions the procedure is completely analogous [7].

Our aim is to establish a similar type of relationship, making use of moment information instead of projections. To this end we need to introduce a few tools.

3 The Stieltjes and Markov integral transforms

The n-variate Stieltjes transform $\mathcal{S}_n^{(f)}(\vec{v})$ of the non-negative function $f(\vec{x})$ defined in $\mathbb{R}_+^n = ([0, +\infty))^n$ is given by

$$\mathcal{S}_n^{(f)}(\vec{v}) = \int_{\mathbb{R}_+^n} \frac{f(\vec{x})}{1 + \vec{x}\cdot\vec{v}}\, d\vec{x} \tag{2}$$

where $\vec{x}\cdot\vec{v} = \sum_{i=1}^n x_i v_i$ for $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{v} = (v_1, \ldots, v_n)$. According to the famous Fourier slice theorem

$$\mathcal{S}_n^{(f)}(\vec{\lambda}u) = \int_{a(\vec{\lambda})}^{b(\vec{\lambda})} \frac{\mathcal{R}_{\vec{\lambda}}^{(f)}(z)}{1 + zu}\, dz \tag{3}$$

where $||\vec{\lambda}|| = 1$ and $[a(\vec{\lambda}), b(\vec{\lambda})]$ denotes the support of the Radon transform $\mathcal{R}_{\vec{\lambda}}^{(f)}$ of the function $f(\vec{x})$.

In other words, the n-variate Stieltjes transform $\mathcal{S}_n^{(f)}$ restricted to the one-dimensional slice

$$S_{\vec{\lambda}} = \left\{ \vec{\lambda} u \in \mathbb{R}^n \mid ||\vec{\lambda}|| = 1, u \in \mathbb{R} \right\}$$

equals the univariate Markov transform

$$\mathcal{M}_1^{\left(\mathcal{R}_{\vec{\lambda}}^{(f)}\right)}(u) = \int_{a(\vec{\lambda})}^{b(\vec{\lambda})} \frac{\mathcal{R}_{\vec{\lambda}}^{(f)}(z)}{1+zu}\, dz$$

A less well-known n-variate Padé slice property asserts that the n-variate Padé approximant $r_{m+k,m}^{(f)}(\vec{x})$ to the function $f(\vec{x})$, as defined in [2, 3], satisfies

$$r_{m+k,m}^{(f)}(\vec{\lambda} z) = r_{m+k,m}^{(f_{\vec{\lambda}})}(z) \qquad m \in \mathbb{N}, k \in \mathbb{Z}$$

where the univariate function $f_{\vec{\lambda}}$ is the restriction of $f(\vec{x})$ to the slice $S_{\vec{\lambda}}$, namely $f_{\vec{\lambda}}(z) = f(\vec{\lambda} z)$, and $r_{m+k,m}^{(f_{\vec{\lambda}})}(z)$ is the classical univariate Padé approximant to the function $f_{\vec{\lambda}}$. Making use of this property, we can show in addition that, if the non-negative function $f(\vec{x})$ is zero outside the unit ball $B(0;1)$

$$\mathcal{S}_n^{(f)}(\vec{v}) = \mathcal{M}_1^{\left(\mathcal{R}_{\vec{\lambda}}^{(f)}\right)}(u) = \lim_{m\to\infty} r_{m+k,m}^{\left(\mathcal{S}_n^{(f)}\right)}(\vec{v}) \qquad \vec{v} = \vec{\lambda} u \qquad k \geq -1 \tag{4}$$

Note that requiring that $f(\vec{x})$ is zero outside $B(0;1)$ is only a matter of scaling. In the new reconstruction algorithm, this Markov and Radon transform shall not be computed explicitly. We only use of moment information. The importance of the relationship (3) is that it is an identity of the same type as (1).

4 Numerical algorithm

How are the above results used in practice, when the input is moment and not Radon information and without the need to compute the Markov integral transform? How does the use of these results compare to (1)?

To answer the first question, we briefly describe the reconstruction of a compact shape $A \subset \mathbb{R}_+^2 \cap B(0,1)$. Let $f(\vec{x})$ be the characteristic function of the set A, in other words $f(\vec{x}) = 1$ for $\vec{x} \in A$ and $f(\vec{x}) = 0$ for $\vec{x} \notin A$. Having available the shape's moments

$$c_{ij} = \int_{\mathbb{R}_+^2} x^i y^j f(x,y)\, dx\, dy = \int_A x^i y^j \, dx\, dy$$

we can compute a Padé approximant for the Stieltjes transform $S_n^{(f)}$ of which the formal series representation is obtained from (2) and is given by

$$\sum_{i,j=0}^{\infty} \binom{i+j}{i} (-1)^{i+j} c_{ij} v^i w^j \qquad \vec{v} = (v,w)$$

This Padé approximant can be evaluated in several points $\vec{v}$. According to (4), the value of the Padé approximant in $\vec{v}$ provides an approximation for $S_n^{(f)}(\vec{v})$. Given a sufficiently large number of evaluations, we are thus able to solve (2) for $f(\vec{x})$. This usually results in solving an ill-posed problem and requires regularization. For this we found the truncated SVD technique to be very successful.

With respect to the second question we note that (1) requires a change from the cartesian to the polar coordinate system. This change creates numerical problems when computing the inverse bivariate Fourier transform, problems which are usually overcome by the use of some interpolation technique. Although the new relation between the Markov and Stieltjes transform uses this change of coordinate system in its description, it is clear from (4) that it can entirely be avoided in the implementation.

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