IMPROVED SPECTRAL ANALYSIS OF NEARBY TONES USING LOCAL DETECTORS

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ABSTRACT

This paper concerns the problem of resolvability power in the frequency domain. The canonical case of interest is to distinguish whether the received noise-corrupted signal is a single-frequency sinusoid or a two-frequency sinusoid, where the amplitudes, phases and frequencies are unknown to the receiver. Using a model-based hypothesis testing approach, we quantify a measure of attainable resolution between sinusoids with nearby frequencies, in the presence of noise. An explicit relationship is derived for the minimum detectable difference between the frequencies of two tones, for any particular false alarm and detection rate, and at a given SNR. An associated algorithm is proposed that produces significantly better performance compared to the standard subspace-based methods like MUSIC and can be effectively used in practice as a postprocessing step for the existing spectral estimation methods.

1. INTRODUCTION

Resolving sinusoidal signals with nearby frequencies has been a very extensively studied problem in array processing. In array processing terms, the problem is related to the case where two incoherent plane waves are incident upon a linear equi-spaced array of sensors [1]. The majority of the past work in this area have been based on the analysis of the second order statistics which is related to the covariance structure of the measured signal [2, 3]. At the same time, a remarkable number of papers have addressed the performance analysis of these subspace methods [1, 4, 5, 6, 7, 8]. The subspace methods (e.g. MUSIC) employ the eigen-decomposition of the estimated autocorrelation matrix into orthogonal signal and noise subspaces [2].

A very common question addressed in the related literature has been to investigate the relationship between resolution and SNR. With respect to this question, most of the relevant papers have focused on the MUSIC algorithm. A rather similar question interests us in this paper: We present a quantitative measure of resolution by addressing the following question:"What is the minimum separation between two nearby tones (maximum attainable resolution) that is detectable at a given signal-to-noise ratio (SNR), and for pre-specified probabilities of detection and false alarm (P_d and P_f)?".

To begin, let the canonical signal of interest be

$$\begin{aligned} s(x;\delta_1,\delta_2) &= a_1 \sin\left(2\pi(f_c-\delta_1)x+\phi_1\right) \\ &+ a_2 \sin\left(2\pi(f_c+\delta_2)x+\phi_2\right) \end{aligned} \quad (1)$$

for the range of $x \in [-\frac{B}{2}, \frac{B}{2}]$, where we consider the two frequencies $f_c-\delta_1$ and $f_c+\delta_2$ to be around a (known or estimated) "center" frequency (f_c). The received signal is a sampled, and noise-corrupted version of (1), i.e.

$$\begin{aligned} f(k;\delta_1,\delta_2) &= s(k;\delta_1,\delta_2)+w(k) \\ &= a_1 \sin\left(2\pi\left(f_c-\delta_1\right)k/f_s+\phi_1\right) \quad (2) \\ &+ a_2 \sin\left(2\pi\left(f_c+\delta_2\right)k/f_s+\phi_2\right)+w(k), \end{aligned}$$

where the sampling frequency is f_s (Hz), assumed to be greater than $2(f_c+\delta_2)$, and the integer index k is in the range $k \in \{-(N-1)/2, \cdots, (N-1)/2\}$, where $N = Bf_s$. The term $w(k)$ is assumed to be a zero-mean Gaussian white noise process with variance σ^2.

According to the so-called Rayleigh criterion [4], the two (equal-amplitude) spectral peaks located at $f_c-\delta_1$ and $f_c+\delta_2$ are unresolvable if $\delta_1+\delta_2 < \frac{1}{B}$. In this paper we are interested in studying this scenario for the signal defined in (2). In particular, we define the term "signals with short observation interval" as the case in which the values of B, δ_1 and δ_2 satisfy the above inequality. In this case the main-lobe of the Fourier transform of the (sum of) two sinusoids is located in the same FFT bin.

The fundamental premise of our approach is to pose the problem of resolution as a hypothesis test. Namely, the corresponding hypotheses for the model in (2) are

$$\begin{cases} \mathcal{H}_0: & \delta_1 = 0 \quad \text{and} \quad \delta_2 = 0 \\ \mathcal{H}_1: & \delta_1 > 0 \quad \text{or} \quad \delta_2 > 0 \end{cases} \quad (3)$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ denote the null hypothesis (one peak is present) and alternative hypothesis (two peaks are present), respectively. Since we consider the case where δ_1 and δ_2 are unknown to the detector, (3) represents a composite (but one-sided) hypothesis testing problem [9, p. 103].

The solution we propose for the hypothesis test in (3) is based upon a locally optimal detection strategy. We assume that the center frequency (f_c) is known beforehand or can be estimated by using one of the existing spectral estimation methods (see [10] for details). However, we assume that the amplitudes, phases and frequency parameters (δ_1 and δ_2) of the sinusoids are unknown to the detector. We should perhaps emphasize that in the subspace methods, the phases of sinusoids are assumed to be independent, uniformly distributed random variables, whereas we treat them as unknown deterministic variables.

The foregoing analysis is useful in two respects: first to develop a locally optimal detection methodology and, second, to establish an explicit performance bound (the result of employing these locally powerful detectors) for the underlying problem. In order to emphasize the practicality of the results, we compare the proposed algorithm against the MUSIC algorithm. We demonstrate that the proposed detectors yield significantly improved performance in distinguishing frequencies of nearby tones.

The organization of this paper is as follows. In Section 2, we develop our detection strategies and characterize their performance. Section 3 presents some results and also comparisons of the proposed method with the existing subspace methods. Finally, in Section 4, we conclude the paper by summarizing the results.

2. DETECTION THEORETIC APPROACH

Since the range of interest for the values of δ_1 and δ_2 is small ($\delta_1,\ \delta_2 < \frac{1}{2B}$), (these representing one wide peak in the frequency domain,) it is quite appropriate for the purposes of our analysis to consider approximating the model of the signal around $(\delta_1, \delta_2) = (0, 0)$. The second order Taylor expansion of (2) about $(\delta_1, \delta_2) = (0, 0)$, with all other variables fixed, is

$$s(k; \delta_1, \delta_2) \approx \alpha_0 p_0(k) + \beta_0 q_0(k) + \alpha_1 p_1(k) + \beta_1 q_1(k) + \alpha_2 p_2(k) + \beta_2 q_2(k) \quad (4)$$

where

$$\begin{aligned}
p_i(k) &= (k/f_s)^i \sin(2\pi f_c k/f_s) \\
q_i(k) &= (k/f_s)^i \cos(2\pi f_c k/f_s) \\
\alpha_0 &= a_1 \cos(\phi_1) + a_2 \cos(\phi_2) \\
\beta_0 &= a_1 \sin(\phi_1) + a_2 \sin(\phi_2) \\
\alpha_1 &= 2\pi(a_1 \delta_1 \sin(\phi_1) - a_2 \delta_2 \sin(\phi_2)) \\
\beta_1 &= 2\pi(-a_1 \delta_1 \cos(\phi_1) + a_2 \delta_2 \cos(\phi_2)) \\
\alpha_2 &= -2\pi^2(a_1 \delta_1^2 \cos(\phi_1) + a_2 \delta_2^2 \cos(\phi_2)) \\
\beta_2 &= -2\pi^2(a_1 \delta_1^2 \sin(\phi_1) + a_2 \delta_2^2 \sin(\phi_2)).
\end{aligned}$$

We elect to keep terms up to order 2 of the above Taylor expansion. This gives a more accurate representation of the signal since in some cases (e.g. $a_1 = a_2$, $\phi_1 = \phi_2$, $\delta_1 = \delta_2$) the first order terms (related to $p_1(k)$ and $q_1(k)$) would be very small or would even vanish. Rewriting (4) in vector form will result in

$$\mathbf{s} \approx \alpha_0 \mathbf{p}_0 + \beta_0 \mathbf{q}_0 + \alpha_1 \mathbf{p}_1 + \beta_1 \mathbf{q}_1 + \alpha_2 \mathbf{p}_2 + \beta_2 \mathbf{q}_2 \quad (5)$$

where for example $\mathbf{p}_1 = \left[p_1\left(-\frac{N-1}{2}\right), \cdots, p_1\left(\frac{N-1}{2}\right)\right]^T$. Now, the hypotheses in (3) appear in the following form:

$$\begin{cases} \mathcal{H}_0 : \mathbf{z} = \alpha_0 \mathbf{p}_0 + \beta_0 \mathbf{q}_0 + \mathbf{w} \\ \mathcal{H}_1 : \mathbf{z} = \alpha_0 \mathbf{p}_0 + \beta_0 \mathbf{q}_0 + \alpha_1 \mathbf{p}_1 + \beta_1 \mathbf{q}_1 + \alpha_2 \mathbf{p}_2 + \beta_2 \mathbf{q}_2 + \mathbf{w} \end{cases}$$

where $\mathbf{z}$ denotes the approximate measured signal model. This leads to a linear model for testing the parameter set θ defined as follows:

$$\begin{aligned}
\mathbf{z} &= \mathbf{H}\theta + \mathbf{w} && (6) \\
\mathbf{H} &= [\mathbf{p}_0 |\ \mathbf{q}_0 |\ \mathbf{p}_1 |\ \mathbf{q}_1 |\ \mathbf{p}_2 |\ \mathbf{q}_2] && (7) \\
\theta &= [\alpha_0\ \beta_0\ \alpha_1\ \beta_1\ \alpha_2\ \beta_2]^T && (8)
\end{aligned}$$

where $\mathbf{H}$ and θ are an $N \times 6$ matrix, and a 6×1 vector, respectively. The corresponding hypotheses are

$$\begin{cases} \mathcal{H}_0 : & \mathbf{A}\theta = \mathbf{0} \\ \mathcal{H}_1 : & \mathbf{A}\theta \neq \mathbf{0} \end{cases} \quad (9)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (10)$$

The hypothesis test in (9) is a problem of detecting a deterministic signal with unknown parameters (θ). Since the probability density function (PDF) under $\mathcal{H}_0$ and $\mathcal{H}_1$ is not known exactly, we form the Generalized Likelihood Ratio Test (GLRT) to obtain the test statistic. GLRT first computes maximum likelihood (ML) estimates of the unknown parameters, and then will use this estimated value to form the standard Neyman-Pearson (NP) detector [11, p.186]. The GLRT for (9) is given by [11, p. 274]

$$T = \frac{1}{\sigma^2} \widehat{\theta}^T \mathbf{A}^T \left[\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right]^{-1} \mathbf{A}\widehat{\theta} \quad (11)$$

where $\widehat{\theta} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{z}$ is the unconstrained maximum likelihood estimate of θ. For any given data set $\mathbf{z}$, we decide $\mathcal{H}_1$ if the statistic exceeds a specified threshold, $T(\mathbf{z}) > \gamma$. The choice of γ is motivated by the level of tolerable false alarm in a given problem. From (11), the performance of this detector is characterize by

$$\begin{aligned}
P_f &= Q_{\chi_4^2}(\gamma) && (12) \\
P_d &= Q_{\chi_4'^2(\lambda)}(\gamma) && (13) \\
\lambda &= \frac{1}{\sigma^2}\theta^T \mathbf{A}^T \left[\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right]^{-1} \mathbf{A}\theta, && (14)
\end{aligned}$$

where $Q_{\chi_4^2}$ is the right tail probability for a Central Chi-Squared PDF with 4 degrees of freedom, and $Q_{\chi_4'^2(\lambda)}$ is the right tail probability for a non-central Chi-Squared PDF with 4 degrees of freedom and non-centrality parameter λ.

Now to compute the required SNR for a given separation, we first compute the value of the non-centrality parameter ($\lambda(P_f, P_d)$) according to the pre-specified P_d and P_f by using (12) and (13). Then the value of noise variance can be obtained from (14). Finally, by defining (the output) SNR as SNR $= \theta^T \mathbf{H}^T \mathbf{H}\theta/\sigma^2$ and replacing the value of noise variance from the previous step, the relation between the parameter set θ and the required SNR will be given by:

$$\text{SNR} = \lambda(P_f, P_d) \frac{\theta^T \mathbf{H}^T \mathbf{H}\theta}{\theta^T \mathbf{A}^T \left[\mathbf{A}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{A}^T\right]^{-1}\mathbf{A}\theta} \quad (15)$$

To gain further insight, we consider a special case of the signal model in (2). We assume that $a_1\delta_1 \approx a_2\delta_2$, which results from a proper choice of the center frequency f_c (See [10]) and simultaneously consider the case where the value of ϕ_1 is close to that of ϕ_2. Consequently after some algebra, by replacing $\frac{N}{f_s}$ with B and by neglecting non-dominant terms, (15) for small δ_1 and δ_2 (i.e. $\delta_1,\ \delta_2 \ll \frac{1}{B}$) will result in

$$\text{SNR} \approx \frac{45}{\pi^4}\frac{\lambda(P_f, P_d)}{B^4}\frac{a_1^2 + a_2^2 + 2a_1a_2}{a_1^2\delta_1^4 + a_2^2\delta_2^4 + 2a_1a_2\delta_1^2\delta_2^2} \quad (16)$$

Equation (16) is representing an explicit relationship one can use to understand the required SNR to achieve a particular resolution level of interest below the Rayleigh limit. It is also informative to compute the required SNR for the case where $\delta_1 = \delta_2 = \delta$, $a_1 = a_2 = 1$. This yields

$$\text{SNR} \approx \frac{45}{\pi^4}\frac{\lambda(P_f, P_d)}{(B\delta)^4} \quad (17)$$

which shows that for the case where the frequencies of the two sinusoids are symmetrically located about the test point, the required SNR is proportional to the inverse of the frequency separation to the power of 4.

3. RESULTS

A plot of (16) is shown in Figure 1 for the case of equal amplitude and for the case of $a_1 = 4a_2$ (In either case, the amplitudes and phases are not known to the detector.). The case of equal amplitudes produces better detection performance, as expected.

As mentioned before, we consider the phases of sinusoids to be unknown deterministic variables. Whereas for subspace detectors, the phase is typically assumed to be a uniformly distributed random variable in $[0, 2\pi]$. Also the "required SNR" computed in (15) is in general a function of the phases

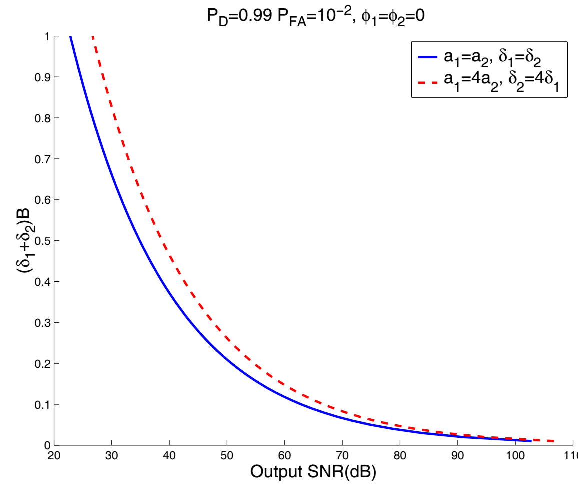

Fig. 1. $(\delta_1 + \delta_2)B$ vs. required SNR for equal and unequal amplitudes.

of the sinusoids. Therefore in order to set up a fair comparison to subspace methods, we average the required SNR over the possible range of ϕ_1 and ϕ_2:

$$\text{SNR}_{\text{avg}} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \text{SNR}\ \ d\phi_1 d\phi_2 \quad (18)$$

where subscript "avg" denotes the averaged value and the integrand (SNR) is the right hand side of (15).

To compare with subspace methods, we simulated the behavior of the MUSIC algorithm for resolving sinusoids with nearby frequencies. In simulation of MUSIC, the signal is declared to be resolvable if the output of MUSIC produces two distinct peaks within an interval around the true frequencies ($f_c \pm \delta$). The simulations for MUSIC are carried out for cases in which either a single snapshot, or multiple snapshots, are available. Naturally, we consider the output SNR in the latter case as the sum of SNR's of each snapshot. We develop two different comparison procedures. First, we compare the performance of MUSIC with the performance of the detector in (11), where we assume that the center frequency f_c, at which we perform the hypothesis test, is known a priori. We have also put forward an alternative (perhaps more practical) scenario, too. In this scenario, the center frequency is first estimated by MUSIC and the proposed detector is then applied, centered at the peak estimated by MUSIC.

The results of these experiments are shown in Figure 2. First, we observe that the proposed detector significantly outperforms MUSIC in both cases (using known or estimated center frequency). More interestingly, we see that the result of the proposed detector with estimated center frequency (provided by MUSIC) is very close to the performance of the same detector with known center frequency,

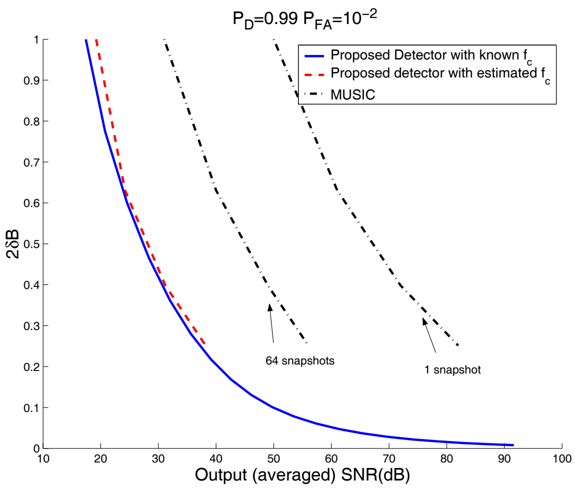

Fig. 2. $2\delta B$ vs. required output SNR for the MUSIC algorithm.

the latter representing the ultimate performance bound. This implies that the MUSIC algorithm does a very promising job in locating the center frequency, which can then be used to further refine the spectral estimate using our proposed approach.

4. CONCLUSION

The problem of interest in this paper has been to carry out a statistical performance analysis for distinguishing whether the received signal is a single-tone or a double-tone signal. We have considered a general case where the amplitudes, frequencies, and phases of sinusoids are unknown to the detector. We consider the case where the two resulted frequency patterns from the sinusoids fall in the same FFT bin (i.e. short observation interval for the signal).

By utilizing a global quadratic approximation, we in fact carried out the analysis in the context of locally optimal detectors, and developed corresponding detection strategies.The question of limits to resolution has been addressed by formulating the following practical question: "What is the minimum detectable frequency difference between two sinusoids at a given signal-to-noise ratio?"

Comparing to existing spectral estimation method, the proposed detection algorithm produces remarkably improved detection of nearby frequencies. As for implementation, a test point candidate for the detector can be first identified by one of the existing spectral estimation methods and then the suggested detector can be applied as a post-processing stage to gain better resolution. The application of such a detector, which uses (for example) MUSIC to estimate the center frequency as the test point, is nearly as effective as applying the proposed detector with a known center frequency.

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