Equivalence and Decomposition in Queueing Systems—A Unified Approach

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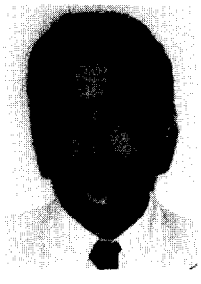
This paper is an attempt to cast in a unified framework several approximation techniques based on equivalence (aggregation) and decompositon that have been applied for the solution of queueing networks in the last decade or so. The main idea in the proposed framework is that it is possible, and useful, to view these techniques as involving two separate steps. Equivalence, the first step, is merely obtaining state equations for a suitably chosen marginal probability, and, as such, is an exact transformation regardless of whether the system is nearly completely decomposable or possesses a product-form solution. The second step, decomposition, is the computation of conditional probabilities introduced through equivalence. This is where approximations enter the picture. Besides providing a simple link between methods such as Norton's theorem, near complete decomposability or the X-model, the proposed framework has merits of its own. It has the potential to facilitate the search for new improved approximations, allows the derivation of bounds on some quantities and constitutes the basis for a family of numerical solution methods for queueing systems.

Keywords: Queueing Models, Approximation Techniques, Equivalence (aggregation) and Decomposition, Marginal and Conditional State Probabilities, Relationship between Approximations, Numerical Methods.



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1. Introduction

Equivalence and decomposition approaches to the solution of queueing networks were introduced close to a decade ago by Courtois [20,21] for weakly coupled subsystems, Avi-Itzhak and Heyman [2] as an approximation in the study of multiprogramming systems, Chandy, Herzog and Woo [17] as approximate analogy to the Norton-Thevenin reduction theorem in linear electrical networks, and by Brandwajn [9,10] with the idea of exact equivalence for a specific class of networks. Since then, these or closely related methods have been applied to a variety of problems such as

* virtual memory and multiprogramming, e.g. [11,26,41,46,38,43,15,34]
* multiprocessor systems, e.g. [33]
* input/output subsystems, e.g. [14,7]
* concurrency, e.g. [27]
* locking, e.g. [1,50]

to quote a few.

What these approaches have in common, is the idea to replace the solution of a single complex system by a set of solutions of simpler (sub)systems, and to combine these "decomposed" solutions by aggregating the original queueing network, i.e., by replacing it with an "equivalent"system.

Despite the time since the introduction of these approaches, and their widespread use, the precise relationship between them seems to remain unclear. Is the short-circuit method of "Norton's theorem" the same as near-complete decomposability or are these two totally different approaches? Is equivalence exactly true for systems

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with product-form solutions and approximately satisfied for nearly completely decomposable systems only, or is it a more general property independent of product form and loose coupling? What is the relationship between the equivalence and decomposition approaches and the "X-model" [28] and related iterative methods [6,30]?

In this paper, we attempt to provide an answer to these questions by proposing a framework for a unified view of these approaches. The main idea in the proposed framework is to regard these techniques as consisting of two separate steps. The first step, referred to as equivalence, is simply a state reduction where some original state description is replaced by a suitably chosen marginal probability. This transformation is exact, independently of near-complete-decomposability or product-form properties. The second step, broadly termed decomposition, is the computation of conditional probabilities invariably introduced through equivalence. An exact computation is usually possible only for specific types of systems, so that these conditional probabilities are often only approximated.

Section 2 of this paper discusses the notion of equivalence. The next section is devoted to the decomposition step. It reviews various methods used and establishes the relationship with the "X-model". In Section 4, we discuss a few issues related to the proposed framework such as bounds on some quantities and efficient numerical solutions for nearly-decomposable systems.

2. Equivalence

Assume we are given a queueing system which, for whatever reason, we want to solve using an equivalence and decomposition approach. Initially, we think of this queueing system in terms of some state description (a vector) $s = (s_1, \ldots, s_i, \ldots, s_k)$. Usually, this state description is such that it is possible, at least in theory, to obtain fully defined state equations for our system with the given parameters.

As a first step, we choose a less detailed, "marginal" state description $s^* = (s_1, \ldots, s_i)$ which contains some initial subset of s. Let $\hat{s} = (s_{i+1}, \ldots, s_k)$ denote the disregarded part of the original state description. The equations for the new reduced state s^* can be obtained by summing the original state equations for s over all values of $\hat{s}$, and using the relationship between state probabilities

$$p(s) = p(s^*)\,p(s \mid s^*) \tag{1}$$

which may be rewritten as

$$p(s) = p(s^*)\,p(\hat{s} \mid s^*), \tag{2}$$

so that

$$p(s^*) = \sum_{\{\hat{s}\}} p(s),$$

with

$$\sum_{\{\hat{s}\}} p(\hat{s} \mid s^*) = \sum_{\{\hat{s}\}} p(s \mid s^*) = 1. \tag{3}$$

As a result of the summation, if $a(s)$ is the coefficient of $p(s)$ in the original state equations, it will be replaced by

$$\sum_{\{\hat{s}\}} a(s)\,p(s \mid s^*)$$

as a corresponding coefficient for $p(s^*)$ in the reduced set of equations. Thus, generally speaking, the coefficients in the new state equations are expressed in terms of parameters of the queueing system under consideration and conditional probabilities of the original state s given the new one s^*.

If we interpret summation to include integration for continuous state variables, the above transformation will apply to any state description whose probability/probability density satisfies a set of linear equations, i.e., the state equations are linear. This includes balance equations for Markovian systems and steady state equations for the embedded Markov chain and supplementary variables methods in non-Markovian systems, as well as transient state equations in Markovian systems. Unless stated otherwise, in the sequel, we shall limit ourselves to steady state considerations.

The view we propose is that the equivalent (or aggregated) system is defined through the set of equations for $p(s^*)$, i.e., for the reduced state description.

This view has a number of consequences. First, equivalence becomes a well defined property. It is exact, independent of whether the original queueing system possesses a product-form solution or weakly interacting subsystems. Second, it is applicable to non-Markovian systems. Indeed, equivalence is nothing more than defining a suitably

chosen marginal probability distribution/density. The reduced state description s^* is chosen so as to simplify the solution of the equations for $p(s^*)$, i.e., for the equivalent system. A third consequence of this view is that the system to which our original system is equivalent is merely an interpretation of equations for $p(s^*)$, a projection onto a queueing system having the same set of equations. In some cases (especially for systems analysed by the embedded Markov chain method), there may be no readily identifiable "physical" equivalent.

The fact that the equivalent system in an interpretation of equations is one reason why this equivalent system may not be unique. As an example, take a set of reduced equations identical to those of an M/M/1/N queue with a state dependent service rate $u(n)$. Depending on the particular initial problem being solved, it may be useful to visualize the equivalent queue as a set of state dependent memoryless delays of duration $n/u(n)$ (cf [42]), or as a queue with a bypass loop as in [32].

Another reason is, of course, that there may be several initial state descriptions s and a number of different marginal state descriptions s^*.

Two simple examples to illustrate some of these ideas follow.

Example 1. Consider a First Come First Served single server queue with Poisson arrivals and hyperexponential service times represented by two exponential service stages (see Fig. 1). The arrival rate is λ, μ_1 and μ_2 are the parameters of the service stages, and q_i is the probability that stage i is chosen by a customer entering service.

Let (n, i) for $n > 0$, and n for $n = 0$ be the initial state description s for this system. Choose $s^* = (n)$, i.e., the total number of customers in the system, as the reduced state. From the balance equations for $p(s)$, we readily obtain equations for the new state description n.

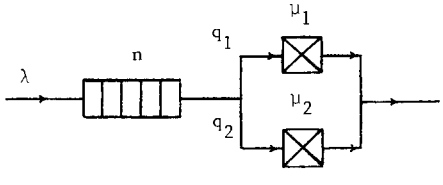

Fig. 1. A hyperexponential queue.

$$p(0) = u(1)p(1),$$

$$[\lambda + u(n)]\,p(n) = \lambda p(n-1) + u(n+1)p(n+1), \quad n = 1, 2, \ldots,$$

where

$$u(n) = \sum_{i=1}^{2} \mu_i p(i|n).$$

$p(i|n)$ is the conditional probability that $i (i = 1, 2)$ is the current service stage given that there are n customers in the system. These equations may be interpreted to represent an M/M/1 queue with state-dependent service rate $u(n)$. Hence, the equivalent network of Fig. 2. The "equivalent" service rate $u(n)$ is expressed in terms of μ_i — parameters of the original system — and $p(i|n)$, conditional probability of the original state description (n, i) given the reduced one, n. In respect to the state vector (n, i), $p(n)$ is a marginal probability distribution.

The equivalence and solution of a generalized class of queues of this type has been the object of recent work by Marie [36,37].

Example 2. Consider the queueing network shown in Fig. 3. It consists of three exponential servers labelled 1, 2 and 3. Denote by n_i the number of customers at server i $(i = 1, 2, 3)$, and by n the total number of customers at servers 1 and 2, i.e., $n = n_1 + n_2$. Denote also by N the constant total number of customers in the network. The service rate of server 3 is $\lambda(n)$, the rates of departures from server 2 (when active) to servers 1 and 3 are β_1 and β_3, respectively. The queueing room at server 2 has a limited capacity m, so that server 1 is blocked whenever n_2 reaches the value m, and resumes service with rate α as soon as n_2 drops below that value.

Let $s = (n, n_2)$ be the initial state description, and choose $s^* = (n)$ to be the reduced description. The equations for $p(n)$ are easily seen to be

$$\lambda(0)p(0) = u(1)p(1);$$

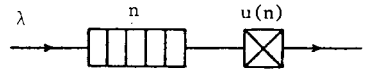

Fig. 2. A visualization of equations.

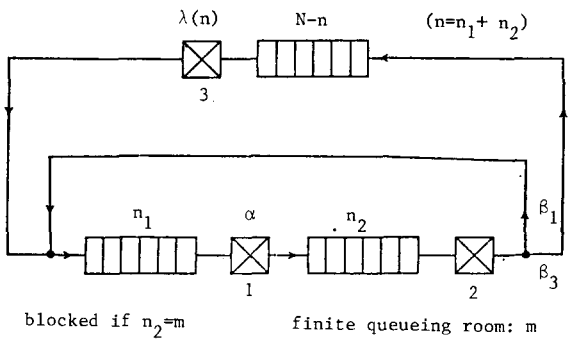

Fig. 3. Queueing network with blocking.

$$[\lambda(n)+u(n)]\,p(n)=$$
$$\lambda(n-1)p(n-1)+u(n+1)p(n+1),$$
$$n=1,\ldots,N-1;$$
$$u(N)p(N)=\lambda(N-1)p(N-1),$$

with

$$u(n)=\beta_3\sum_{n_2=1}^{\min(m,n)}p(n_2\,|\,n).$$

$p(n_2\,|\,n)$ is the conditional probability that there are n_2 customers at server 2 given the total number of customers at servers 1 and 2. We choose to interpret these equations as representing a queueing system in which servers 1 and 2, together with the feedback loop around them, have been replaced by N memoryless delay servers, each with instantaneous service rate $u(n)/n$ when n of these servers are busy (see Fig. 4). Note that only one conditional probability is actually needed in this example to express $u(n)$ since the latter can be rewritten as

$$\beta_3[1-p(n_2=0\,|\,n)].$$

In summary, we propose to view equivalence as an exact state reduction where the paremeters of the equivalent (reduced) system are expressed in terms of parameters of the original system and of conditional probabilities of the original state description given the new reduced one.

In previous work, the idea of exact equivalence is mostly confined to specific types of queueing networks. Chandy, Herzog and Woo [17], and more recently, Balsamo and Iazeolla [5] consider queueing networks with product-form solutions within the scope of Norton's theorem. Vantilborgh [52] discusses exact aggregation in product-form networks in general mathematical terms based on the approach to nearly completely decomposable

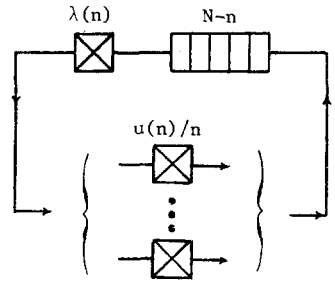

Fig. 4. A possible equivalent network.

systems used by Courtois [20,23]. Marie [36] also uses the notion of exact state reduction. Stewart and Zeiszler [48] study exact equivalence in networks of Coxian queues [24]. Brandwajn [9,10] considers the idea of exact eqivalence in generalized central server queueing networks, and extends that notion to other queueing systems in [13].

As we have seen, equivalence introduces the conditional probabilities $p(\hat{s}\,|\,s^*)$ which enter into the expression of parameters of the reduced system. In queueing networks with limited state dependencies, often encountered in practice, only a small subset of all $p(\hat{s}\,|\,s^*)$ is needed in the parameters of the equivalent system. If only the marginal distribution $p(s^*)$ is of interest in the problem being solved, it suffices to evaluate this small subset of conditional probabilities (cf [5]); in other cases, the whole set of $p(\hat{s}\,|\,s^*)$ may be required. We propose to regard the computation of the conditional probabilities $p(\hat{s}\,|\,s^*)$ as a separate step: the decomposition part of the approach. This computation is often approximate, and, in the next section, we consider some of the methods employed and the relationship between them.

3. Decomposition

An aggregation approach is of practical interest only if the parameters of the aggregate can be obtained from the analysis of a system simpler than the original one. Several methods for determining such a simpler ("decomposed") system have been used (cf [19,51,4]), and we broadly refer to them by the term decomposition. In our unified framework, we propose to view these methods as ways to evaluate, in general approximately, the conditional probabilities $p(\hat{s}\,|\,s^*)$ introduced in the equivalence step. This is to say that, in our view,

the various approaches differ mainly in the way these conditional probabilities are approximated.

One such an approach is based on the theory of nearly completely decomposable systems [47], and has been introduced to queueing networks by Courtois [20]. Loosely speaking, in this approach $p(\hat{s}\,|\,s^*)$ is approximated by setting to zero the rates of transitions that change s^*, and analysing the remaining transitions for $\hat{s}$ independently for each vector s^*. When s^* corresponds to aggregated states of weakly interacting subsystems, this results in the conditional probability $p(\hat{s}\,|\,s^*)$ being approximated by the stationary behaviour of the subsystems in isolation, with the condition s^* maintained indefinitely.

In the Norton's theorem method [17,5], decomposition is obtained by letting uniformly to infinity rates of transitions that change s^*. This results in "shorting out" (or "short-circuiting") parts of the system, and the steady state probability of $\hat{s}$ in thus decomposed system, with s^* maintained fixed, is used as an approximation for the conditional probability $p(\hat{s}\,|\,s^*)$. The procedure proposed by Balsamo and Iazeolla in [5] for product-form networks of the BCMP class [8] avoids the explicit computation of all $p(\hat{s}\,|\,s^*)$, and yields only the state dependent parameters of the equivalent system. Note, however, that knowing the whole set of $p(\hat{s}\,|\,s^*)$, it is possible to obtain the probability distribution/density for the full original state description as

$$p(s) = p(s^*)\,p(\hat{s}\,|\,s^*).$$

Whether this is desirable depends on the problem being solved.

The short-circuit approach is known to actually produce exact results for BCMP class [8] networks [5], but not necessarily all other product-form queueing systems. The general conditions for exact solution with near complete decomposability in exponential queueing networks are considered in [52]. Vantilborgh, Garner and Lazowska [53] establish a link between the two approaches for networks with clusters of strongly interacting servers. It is important to note that these two approaches generally do not yield the same results. A simple example to that effect follows. In practice, it has been the author's experience that, when the two methods differ, near-complete-decomposability tends to produce more accurate results in systems with weak interactions, but the "short-circuit" method tends to be more robust in general situations.

Example 3. Consider again the queueing system with blocking of Fig. 3 and the equivalence with respect to n, the total number of customers at servers 1 and 2, studied in Example 2. The conditional probability introduced is $p(n_2\,|\,n)$.

Let

$$x(k) = \begin{cases} 0, & \text{if } k = 0; \\ 1, & \text{otherwise.} \end{cases}$$

The transitions that change the condition variable n are arrivals from, and departures to, server 3 which occur with rates $\lambda(n)$ and $x(n_2)\beta_3$, respectively. Neglecting these transitions (setting their rates to 0), leaves us with the simple two-server network within finite queueing room and blocking shown in Fig. 5.

The stationary probability that there are n_2 customers at server 2 in this network, denoted by $p_{ND}(n_2\,|\,n)$, is the near-decomposability approximation to $p(n_2\,|\,n)$. It is easily seen to be

$$p_{ND}(n_2\,|\,n) = G_{ND}(n)(\alpha/\beta_1)^{n_2},$$

$$n_2 = 0, 1, \ldots, \min(m, n),\ n = 1, \ldots, N;$$

where $G_{ND}(n)$ is a normalization constant.

In the Norton's theorem short-circuit method, we short out server 3, and evaluate the throughput along the short-circuit line to obtain the service rate of the equivalent server. The resulting network is shown in Fig. 6. Denote by $p_{SC}(n_2\,|\,n)$ the probability of n_2 users at server 2 with a total of n users in this network. The throughput through the short-circuit path is

$$\beta_3 \sum_{n=1}^{\min(m,n)} p_{SC}(n_2\,|\,n) = \beta_3\left[1 - p_{SC}(0\,|\,n)\right].$$

Comparing this expression with the service rate $u(n)$ in the reduced system of equations in Exam-

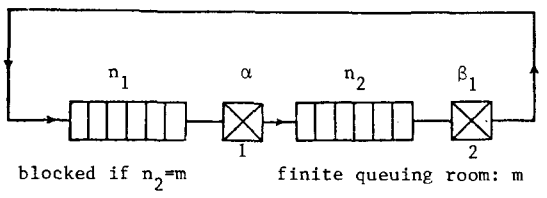

Fig. 5. Subnetwork for near-decomposability approach.

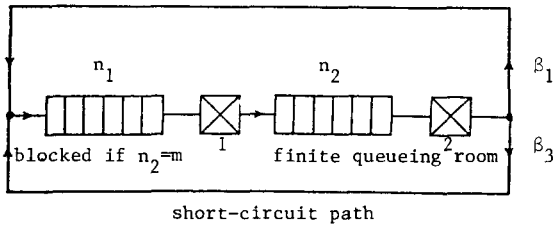

Fig. 6. Network for Norton's theorem.

ple 2, we see that the short-circuit method is indeed an approach for evaluating the conditional probabilities introduced through equivalence. $p_{SC}(n_2 \mid n)$ is readily obtained as

$$p_{SC}(n_2 \mid n) = G_{SC}(n)\left[\alpha/(\beta_1 + \beta_3)\right]^{n_2},$$

$$n_2 = 0, 1, \ldots, \min(m, n),\ n = 1, \ldots, N;$$

where $G_{SC}(n)$ is a normalization constant.

Simple comparison of expressions for the two approximations shows that they do not yield the same results for the system under consideration.

A third approach, employed by Kurinckx and Pujolle [33], is to use the marginal probability $p(\hat{s})$ as an approximation to $p(\hat{s} \mid s^*)$. The approach is directly applicable to systems in which the set of feasible states $\hat{s}$ is independent of s^*, as might be the case e.g. for an open system described by numbers at each service station. Although at first it may seem quite gross, the approximation finds its justification in the fact that, in open product-form networks, stations do behave as if they were independent (cf [31]). Kurinckx and Pujolle claim, for the systems they investigate in [33], good results in almost all the cases, although their algorithm, as stated, appears flawed (see Appendix).

Other related methods which may be viewed as based on approximating the conditional probabilities introduced through equivalence include the iterative method of Marie [35] and the Heuristic Aggregation Method of Neuse and Chandy [40]. Note that requiring a specific form for the "equivalent" (composite) server, as is the case in these methods, results in constraining the conditional probability approximation to a particular form, e.g., a function of only the total number in a subsystem, even though the reduced state description s^* contains also other state variables.

Let us now, for a moment, restrict our attention to a Markovian queueing network in which two subnetworks, A and B, have been distinguished. We define an initial state description for the network in terms of state descriptions for the two subsystems $s = (s_A, s_B)$, where the state description for subnetwork B consists of the total number of customers in that subsystem and some other state vector s_i: $s_B = (n, s_i)$. Additionally, we assume that parameters in subnetwork A are independent of the internal state of subnetwork B, s_i, and that n undergoes only single-step transitions.

Choose $s^* = (s_A, n)$ as the reduced state description for equivalence. Denote by $u(s^*)$ the rate of transitions which decrease n in the resulting balance equations for $p(s^*)$. The equations for $p(s^*)$ can be interpreted to represent a system consisting of subnetwork A and a memoryless state dependent delay with rate $u(s^*)/n$, i.e., subnetwork B of the original queueing network has been replaced by a delay server. Note that $W(s^*) = n/u(s^*)$ can be interpreted as the average time a customer spends in subsystem B when the state of the network is s^*, and it is expressed in terms of network parameters and conditional probabilities $p(s_i \mid s^*)$.

Let W denote the average sojourn time in subsystem B. It seems reasonable to use W as an approximation for $W(s^*)$. This results in representing a subsystem by the average time spent in it, and is precisely the idea of the "X-model" [28]. The X-model approach thus becomes another (implicit) way to approximate the conditional probabilities $p(\hat{s} \mid s^*)$ that enter into the expression of parameters of the equivalent system. This, and the closely related method of surrogate delays, proved succesful in practice cf, e.g., [6,30].

By regarding aggregation approaches as consisting of two separate steps: exact equivalence, and generally approximate decompositon, we gain a unified view of various approximations and a clear idea where and what approximations are actually introduced. Additionally, as we discuss in the next section, the proposed framework may facilitate the search for new approximations, and also constitutes the basis for a family of numerical solution methods.

4. Further discussion

As mentioned in previous sections, we view aggregation approaches as performing a dichotomy of an initial, detailed state description s into

two parts $s = (s^*, \hat{s})$ according to the relationship for state probabilities ((2))

$$p(s) = p(s^*)p(\hat{s} \mid s^*).$$

Equivalence is then an interpretation of the equations for the marginal probability $p(s^*)$. These equations are (at least in principle) perfectly well defined and their coefficients involve the conditional probabilities $p(\hat{s} \mid s^*)$ which are obtained by decomposition, usually only approximately.

Note that whenever the formal solution for $p(s^*)$ is known, it can be used, together with relation (2), in the set of equations for the original state description s to obtain the exact equations that govern the conditional probabilities $p(\hat{s} \mid s^*)$. Knowing such equations, it is possible to analyse qualitatively a given approximate solution by simply substituting it in the exact equations and examining the noncancelled remainders. More importantly, the form of the exact equations for the conditional probabilities may be of help in searching for improved approximations, better suited to the particular problem being solved. This is illustrated in the example that follows. For queueing networks with general servers, Sauer and Chandy [43] and Sevcik et al. [45] consider improved approximations obtained by including second moments in the definition of a general server to which a subnetwork is aggregated. Note that, with appropriate state descriptions, this approach can be cast into the proposed framework.

Example 4. *Derivation of a new approximation.* Consider a system composed of two exponential queues with coupled servers and limited queueing rooms (see Fig. 7). Denote by n_i $(n_i = 0, \ldots, N_i)$ the number of customers at server i $(i = 1, 2)$ in this system. Let λ_i be the rate of arrivals to queue i assumed to come from a Poisson source. The service rate of queue i $(i = 1, 2)$ is μ_i^* whenever the other queue is empty, and becomes μ_i when there are customers at the other queue.

Let $s = (n_1, n_2)$ be four initial state description for the queues under consideration. Note that this system is known not to possess a product-form solution for the joint probability distribution, $p(n_1, n_2)$. Choose $s^* = (n_1)$ as the reduced state description for equivalence, and denote by $p(n_2 \mid n_1)$ the conditional probability that there are n_2 customers at server 2 given n_1 customers at server 1. Formally, the solution for the marginal

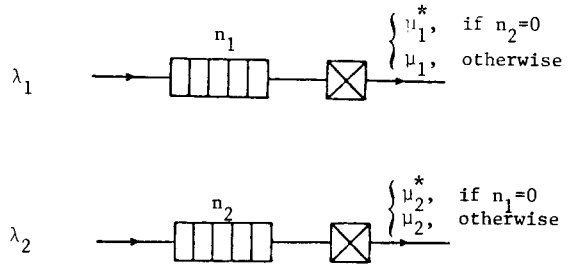

Fig. 7. Two coupled queues.

probability $p(n_1)$ can be readily obtained as

$$p(n_1) = G\lambda_1^{n_1} / \prod_{i=1}^{n_1} u_1(i), \quad n_1 = 0, \ldots, N_1,$$

where

$$u_1(n_1) = \mu_1^* p(0 \mid n) + \mu_1 [1 - p(0 \mid n)],$$

and G is a normalization constant.

For notational convenience, let

$$\lambda_i(n_i) = \begin{cases} \lambda_i, & \text{for } n_i = 0, \ldots, N_i - 1; \\ 0, & \text{for } n_i = N_i; \end{cases}$$

and

$$\mu_i(n_i) = \begin{cases} \mu_i^*, & \text{if } n_j = 0; \\ \mu_i, & \text{if } n_j > 0, \end{cases} \quad i, j = 1, 2, \ i \neq j.$$

The exact equations for the conditional probability $p(n_2 \mid n_1)$ can be written as

$$\begin{aligned} &p(n_2 \mid n_1)[\lambda_1(n_1) + \lambda_2(n_2) + \mu_1(n_2) + \mu_2(n_1)] \\ &\quad = \lambda_2 p(n_2 - 1 \mid n_1) + \mu_2(n_1) p(n_2 + 1 \mid n_1) \\ &\qquad + u_1(n_1) p(n_2 \mid n_1 - 1) \\ &\qquad + \lambda_1(n_1)\mu_1(n_2) p(n_2 \mid n_1 + 1)/u_1(n_1 + 1), \\ &n_2 = 0, \ldots, N_2 - 1; \quad n_1 = 0, \ldots, N_1; \end{aligned}$$

with impossible terms assumed to vanish.

Suppose we are interested in an iterative solution of this set of equations so as to obtain $p(n_2 = 0 \mid n_1)$ necessary to evaluate $u_1(n_1)$ and, hence, $p(n_1)$. Let us consider these equations in the order of increasing n_1. Note that, for any value of $n_1 < N_1$, the equations involve the conditional probability for $n_1 + 1$ through the term $\lambda_1\mu_1(n_2)p(n_2 \mid n_1 + 1)/u_1(n_1 + 1)$. This means that we would have to store the values of $p(n_2 \mid n_1)$ for all n_1 from one iteration to another.

Let us therefore assume that the ratio

$$p(n_2 \mid n_1 + 1)/u_1(n_1 + 1)$$

stays relatively constant for neighboring values of n_1, i.e.,

$$p(n_2 \mid n_1 + 1)/u_1(n_1 + 1) \simeq p(n_2 \mid n_1)/u_1(n_1).$$

This approximation removes the dependence on the "next" value of n_1, and thus reduces the storage requirements of an iterative solution to the $p(n_2 \mid n_1)$ for only two values of n_1: the one being considered, and the one just computed $(n_1 - 1)$. From the equations for $p(n_2 \mid n_1)$, it is clear that any errors introduced by this approximation should be small if λ_1 is small since it is the weighting factor of the approximate ratio. Note that no errors are introduced when the system has a product-form solution ($\mu_i^* = \mu_i$) as the two ratios are then exactly equal.

The above approximation has been derived by considering directly the exact equations for the conditional probabilities introduced through equivalence. A numerical example has been included in the Appendix to illustrate its accuracy.

As shown in [13], exact equations for the conditional probabilities can be used to derive bounds on selected conditional probabilities and, hence, on parameters of equivalent servers for systems in which these parameters involve only a small subset of the conditional probabilities. A simple example is included in the Appendix. While such bounds may also serve to gauge the accuracy of a given approximate solution, their derivation is unrelated to the values of system parameters. They are thus quite different in nature from the general bounds for the error of the near-complete decomposability aproach obtained by Courtois [22] and Stewart [49]. As pointed out by Stewart (p. 273 [49]), error bounds cast in terms of norms (as is the case in [22,49]) tend to be pessimistic, and, also, difficult to interpret in terms of the components of the vector thus bounded. Error bounds for this approximation method have also been studied by Zarling [54].

Because of large discrepancies in the values of coefficients, nearly completely decomposable systems are difficult to solve with many traditional numerical methods (cf [25]). It is interesting that relation (2) may be used as a basis for fast converging iterative methods particularly well suited to nearly completely decomposable systems, and applicable to more general problems as well. The underlying idea is to partition the state vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_k)$ into some number l of nonoverlapping subsets $\boldsymbol{s}_j$ according to

$$\begin{aligned} p(\boldsymbol{s}) &= p(\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_l) \\ &= p(\boldsymbol{s}_1 \mid \boldsymbol{s}_2, \ldots, \boldsymbol{s}_l)\, p(\boldsymbol{s}_2 \mid \boldsymbol{s}_3, \ldots, \boldsymbol{s}_l) \\ &\quad \ldots p(\boldsymbol{s}_{l-1} \mid \boldsymbol{s}_l)\, p(\boldsymbol{s}_l), \end{aligned} \tag{4}$$

in such a way that the rates of transitions that change the condition in the conditional probabilities are as small as possible compared with the rates of transitions changing the non-condition state vector. In practice, the iterative methods proposed differ in the number of subsets into which the state vector is divided and the particular computation scheme used to solve the subsets. Dodd et al. [25] and Mueller [39] use a partition into two subsets, while Brandwajn [12] uses a partition into individual state variables.

We conjecture that numerical methods which use (4), i.e., operate on conditional and marginal rather than joint probabilities, tend to be inherently better behaved numerically. The reason why this would be so is that, through (4), the state space is partitioned into subspaces independently constrained via their normalization conditions

$$\sum_{\{\boldsymbol{s}_j\}} p(\boldsymbol{s}_j \mid \boldsymbol{s}_{j+1}, \ldots, \boldsymbol{s}_l) = 1 \quad \forall\, (\boldsymbol{s}_{j+1}, \ldots, \boldsymbol{s}_l).$$

This reduces the number of elements on which errors can accumulate. Note also that the values of the individual elements handled are larger than when dealing directly with a joint probability since the latter is a product of the former.

As a final point in this section, we note that the view of equivalence adopted in this paper is useful in studying properties of general queueing networks. As an example, consider the problem of finding a product-form queueing network model in which the marginal queue distributions of each station is identical to that of a given arbitrary general closed queueing network, treated by Balbo and Denning in [3].

Assume the network has N customers of a single class, and, for simplicity, $M = 2$ service stations. We define the state of the network as $\boldsymbol{s} = (\boldsymbol{n}, \boldsymbol{a})$, where $\boldsymbol{n} = (n_1, \ldots, n_M)$ is the vector of numbers of customers at each station, and $\boldsymbol{a}$ is the vector of whatever additional variables are needed to write the equilibrium equations (supplementary variable, stage number for Coxian representation [24]...). Let $\mu_i(\boldsymbol{s})$ be the given instantaneous service rate of station i. Barring "special effects"

like batch departures, thresholds, etc, it is not difficult to show that the marginal queue distribution, $p_i(n_i)$, satisfies a relationship of the form

$$p_i(n-1)\lambda_i(n-1) = p_i(n)u_i(n),\ n = 1,\ldots,N. \tag{5}$$

$u_i(n)$, the rate of departures from station i with n customers, is given by

$$u_i(n) = \sum_{\substack{\{s\} \\ n_i = n}} \mu_i(s)p(s \mid n_i). \tag{6}$$

$\lambda_i(n)$ is the arrival rate to station i generated by the remainder of the network; it also is expressed in terms of the conditional probabilities $p(s \mid n_i)$.

It results from (5) and (6) that we can indeed replace the service stations in our network by queue dependent servers so as to obtain a product-form network with the same marginal distributions $p_i(n_i)$ as in the original network. In our case, $u_i(n_i)$ should be taken as the service rate for station i in the product-form equivalent. (6) shows, however, that the "equivalent" $u_i(n_i)$ depend, in general, on the state of the original network. This suggests that values estimated for $u_i(n_i)$ from a set of measurements could only be used in a predictive study if the state dependence of service stations is inherently limited to their local queue, i.e., if the original instantaneous service rate is a function of n only ($\mu_i(s) = \mu_i(n_i)$). Note that a similar conclusion is discussed in [16] for an M/G/1 queueing system.

5. Conclusions

We have proposed a framework for aggregation and decomposition approaches. The idea is to distinguish two steps. Equivalence, the first step, is a state reduction where an initial detailed state description is replaced by a reduced one. The resulting equations for the marginal probability distribution/density exactly define the "equivalent" system. The parameters of the latter involve original system parameters and conditional probabilities of the detailed state description given the reduced one. Decomposition, the second step, is the computation, often approximate, of these conditional probabilities by analysing systems simpler than the original one.

This view of equivalence holds for original state descriptions whose probability/probability density satisfies a set of linear equations: transient and balance equations for Markovian systems as well as steady state equations for the embedded Markov chain and supplementary variables methods in non-Markovian systems. Note that there is no need to distinguish systems with classes of customers since their state description falls into this category.

From the proposed framework, we gain a unified view of several practical approaches, with a clear idea where and what approximations are used. Additionally, this framework may facilitate the search for new approximations, and it constitutes the underlying basis for a few iterative numerical methods well suited to the solution of nearly decomposable systems. It may also be helpful in studying some general properties of queueing systems.

6. Appendix

6.1. The algorithm of Kurinckx and Pujolle [33]

In the approximate general method of Kurinckx and Pujolle (p. 310 of [33]), the expressions for the fictive arrival and service rates appear ill formed in that the sums are not guaranteed to be finite. Consider a simple case of a two queue open Jacksonian network [29] with constant rates. From formula (2) in [33], the fictive service rate for server 1 becomes

$$\sum_{n_2=0}^{\infty} \mu_1 p(n_1 \mid n_2),$$

where μ_1 is the service rate of server 1. Since we have in this system $p(n_1 \mid n_2) = p(n_2)$, the above expression is infinite. We note that arrival and service rates obtained from exact equivalence would be similar to (1) and (2) in [33] but expressed in terms of $p_i(\vec{n} \mid n_i)$ and not $p_i(n_i \mid \vec{n})$. In our example, the service rate for server 1 would then become

$$\sum_{n=0}^{\infty} \mu_1 p(n_2 \mid n_1) = \mu_1.$$

This is because we have

$$\sum_{\vec{n}} p_i(\vec{n} \mid n_i) = 1, \quad \text{for all } n_i,$$

while no such relationship holds for the conditional probabilities used in [33].

6.2. Numerical example for the equal ratio approximation

Consider the system with two coupled queues discussed in Example 4 with the following parameter values.

$N_1 = 10,\ N_2 = 5,\ \lambda_1 = 1,\ \lambda_2 = 10,$

$\mu_1 = 2,\ \mu_1^* = 4,\ \mu_2 = 30,\ \mu_2^* = 60$

We compare in Table 1 the values for $p(n_2 = 0 \mid n_1)$ obtained using the equal ratio approximation with the exact values for these quantities.

We observe that the equal ratio approximation, derived directly from the exact equations for the conditional probabilities, yields values within .5 percent of the exact solution for this problem. Note that the parameter values are such that changes in the state of queue 2 are much more frequent than those in the state of queue 1. Nonetheless, the approximate solution using near-decomposability can be seen to be overall less accurate for this example.

6.3. Example of bounds for conditional probabilities

Consider once again the queueing network with blocking of Fig. 3 and the equivalence discussed in Example 1. In order to describe the behaviour of the equivalent system, only one conditional probability is needed: $P(0 \mid n)$, the conditional probability that server 2 is idle given the total number of customers at servers 1 and 2. Let us consider the equation for $p(0 \mid n)$

$$-[\lambda(n)+\alpha]\,p(0 \mid n)+\beta_1 p(1 \mid n)$$
$$+\beta_3[1-p(0 \mid n)]\,p(0 \mid n-1)$$
$$+\lambda(n)p(1 \mid n+1)/[1-p(0 \mid n+1)]=0,$$
$$n=1,\ldots,N.$$

Note that, for $n = 1,\ldots, N$,

$$p(1 \mid n)/[1-p(0 \mid n)] \le 1,$$

and $p(1 \mid n) \le 1 - p(0 \mid n)$.

Let $p^*(0 \mid n)$ be an upper bound for $p(0 \mid n)$. Using the above inequalities, we readily obtain the following simple recurrence for the set of $p^*(0 \mid n)$

$$p^*(0 \mid n) = [\beta_3 p^*(0 \mid n-1)+\beta_1+\lambda(n)]$$
$$/[\alpha+\beta_3 p^*(0 \mid n-1)+\beta_1+\lambda(n)],$$
$$n=1,\ldots,N,$$

where

$$p^*(0 \mid 0) = p(0 \mid 0) = 1.$$

Hence, we immediately derive a lower bound for the service rate $u(n)$ in the equivalent network

$$u(n) \ge \beta_3[1-p^*(0 \mid n)], \quad n=1,\ldots,N.$$

In a very similar way one can derive bounds for other queueing problems. The tightness of these bounds depends, of course, on the particular system and parameter set considered. Generally, we think the results tend to be non-trivial. Included in Table 2 is an example of the numerical values obtained for the system with two coupled queues

Table 1

n_1:0	1	2	3	4	5		9	10
exact value								
0.832	0.680	0.657	0.653	0.652	0.652	...	0.652	0.645
equal ratio approximation								
0.835	0.682	0.658	0.653	0.652	0.652	...	0.652	0.646
near-decomposability approximation								
0.833	0.668	0.668	0.668	0.668	0.668	...	0.668	0.668

Table 2

n_1:0	1	2	3	4	5		9	10
simple upper bound								
0.859	0.767	0.767	0.767	0.767	0.767	...	0.767	0.762

of Example 4 and the same parameter values as in the preceding numerical example where the exact value for the conditional probabilities $p(0 \mid n_1)$ can be found.

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