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$\top\top$-closed relations and admissibility

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While developing a method for reasoning about programs, Pitts defined the $\top\top$-closed relations as an alternative to the standard admissible relations. This paper reformulates and studies Pitts's operational concept of $\top\top$-closure in a semantic framework. It investigates the nontrivial connection between $\top\top$-closure and admissibility, showing that $\top\top$-closure is strictly stronger than admissibility and that every $\top\top$-closed relation corresponds to an admissible preorder.

1. Introduction

Reynolds's analysis of parametric polymorphism is based on relations and constructions on relations (Reynolds 1983). Wadler and others have shown how this analysis yields proof principles for polymorphic programs (Wadler 1989; Mairson 1991; Abadi *et al.* 1993; Plotkin and Abadi 1993; Plotkin 1993). Although Reynolds allowed a large class of relations in his work, restrictions are essential for soundness in languages with recursion. A common restriction is to consider only the admissible relations. (The next section reviews the definition of admissibility.) For example, Wadler suggested that the use of admissible relations would allow the extension of his results to a language with polymorphism and recursion (Wadler 1989), and Plotkin relied on admissible relations in developing a logic for such a language (Plotkin 1993).

Other well-behaved classes of relations can be adopted too. In particular, Pitts has recently proposed the $\top\top$-closed relations (Pitts 2000). Relying on these relations, Pitts obtained a useful proof method where questions of admissibility have to be treated only implicitly or not at all. Many of the same theorems can be derived with admissible relations and with $\top\top$-closed relations. A partial explanation for this overlap is that many useful relations can be constructed from functions, and that these relations are both admissible and $\top\top$-closed (see section 3).

In this paper we study the $\top\top$-closure condition, comparing it to admissibility. It is not hard to show that every $\top\top$-closed relation is admissible. On the other hand, not every admissible relation is $\top\top$-closed (see section 5). Nevertheless, we characterize $\top\top$-closed

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relations in terms of admissibility. As we prove in section 4, the $\top\top$-closed relations are induced by certain admissible preorders.

Pitts's work is syntactic, and emphasizes operational methods over denotational or logical methods; $\top\top$-closure and similar concepts seem to appear rather easily in syntactic approaches (Birkedal and Harper 1997; Pitts and Stark 1998). In contrast, admissibility is sometimes hard to define and use in syntactic settings.

The setting for this paper is semantic. We give a semantic analogue for Pitts's definitions; for example, we replace sequences of evaluation contexts (called frame stacks) with strict and continuous functions. The translation into a semantic framework simplifies the comparison with admissibility. It also enables us to eschew subtle questions of syntactic definability. Thus, it permits a more direct approach. Still, the semantic notions remain close to their syntactic counterparts.

The next section introduces some notations and definitions. Section 3 considers relations induced by functions. Section 4 characterizes $\top\top$-closed relations and section 5 then compares $\top\top$-closure with admissibility. Section 6 concludes. This paper does not assume familiarity with Pitts's work and is essentially self-contained; however, it does not review applications developed in previous papers.

2. Notation and definitions

A *partial order* is a set A with a binary, reflexive, transitive, antisymmetric relation $\sqsubseteq$; we often identify the partial order with A or with $\sqsubseteq$. A partial order is *complete* if it has a least element $\perp$ and a least upper bound $\sqcup x_i$ for each infinite chain $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \ldots$. A *relation* R between two sets A and B is a subset of $A \times B$. As usual, we write $a\ R\ b$ for $(a, b) \in R$. Next we define admissibility for relations on complete partial orders.

— Suppose that A and B are partial orders with least elements $\perp_A$ and $\perp_B$. A relation R between A and B is *strict* when $\perp_A\ R\ \perp_B$.
— Suppose that A and B are complete partial orders. A relation R between A and B is *inductive* (or *chain-closed*) when, for every chain $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots$ in A and every chain $b_0 \sqsubseteq b_1 \sqsubseteq b_2 \ldots$ in B, if $a_i\ R\ b_i$ for all i then $\sqcup a_i\ R \sqcup b_i$
— A relation is *admissible* when it is both strict and inductive.

Given two sets, A and B, we often work with families of four relations $T_{A,A}$, $T_{A,B}$, $T_{B,A}$, and $T_{B,B}$ respectively between A and A, between A and B, between B and A, and between B and B. In those situations, we may equivalently define a single relation T on the disjoint union $A \oplus B$ of A and B. We say that T is strict (or inductive, or admissible) if all four relations $T_{A,A}$, $T_{A,B}$, $T_{B,A}$, and $T_{B,B}$ are strict (or inductive, or admissible, respectively).

We write $\mathcal{O}$ for the partial order with elements $\top$ and $\perp$, and with the order $\perp \sqsubseteq \top$. If A is a partial order with a least element $\perp$, and f a function from A to $\mathcal{O}$, then f is *strict* when $f(\perp) = \perp$. If A is a complete partial order, then f is *continuous* when, for every chain $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots$ in A, $f(\sqcup a_i) = \sqcup f(a_i)$.

Suppose that A and B are complete partial orders. We write $A \to_\perp \mathcal{O}$ for the set of strict, continuous functions from A to $\mathcal{O}$. Given a relation R between A and B, we define a relation $R^\top$ between $A \to_\perp \mathcal{O}$ and $B \to_\perp \mathcal{O}$, as follows:

$$f\ R^\top\ g \ \triangleq\ \forall (a,b) \in R.\ f(a) = g(b)$$

and a relation $R^{\top\top}$ between A and B, as follows:

$$a\ R^{\top\top}\ b \ \triangleq\ \forall (f,g) \in R^\top.\ f(a) = g(b)$$

A relation R between A and B is $\top\top$-*closed* when $R = R^{\top\top}$. This definition of $\top\top$-closure is a semantic version of Pitts's operational definition of $\top\top$-closure.

3. Relations from functions

In reasoning about programs, it is common to use relations that arise as the graphs of functions (see for example (Wadler 1989; Bainbridge *et al.* 1990; Abadi *et al.* 1993; Plotkin and Abadi 1993)). When h is a function from A to B, we write $\langle h\rangle$ for the graph of h, that is, for the relation between A and B such that $a\ R\ b$ if and only if $h(a) = b$.

For reasoning in languages with recursion, the use of the graph of an arbitrary function h need not be sound. If h is a strict and continuous function, however, then $\langle h\rangle$ is obviously an admissible relation. Here we prove that if h is a strict and continuous function, then $\langle h\rangle$ is also a $\top\top$-closed relation. This result is the analogue to one of Pitts's syntactic lemmas, which is phrased in terms of frame stacks rather than functions. However, the proof below appears to be substantially different from Pitts's proof. We give it mainly because of the importance of function graphs and because it serves as a small introduction to the harder proofs of the next section.

Proposition 1. Suppose that A and B are complete partial orders and that h is a function in $A \to_\perp B$. Then $\langle h\rangle$ is a $\top\top$-closed relation between A and B.

Proof. We show that if $a\ \langle h\rangle^{\top\top}\ b$ then $a\ \langle h\rangle\ b$, by contradiction. We assume that $a\ \langle h\rangle\ b$ is false, so $h(a) \neq b$, and we construct two functions f and g such that $f\ \langle h\rangle^\top\ g$ but $f(a) \neq g(b)$, so $a\ \langle h\rangle^{\top\top}\ b$ is false too.

Since $h(a) \neq b$, either $h(a) \not\sqsubseteq b$ or $b \not\sqsubseteq h(a)$. We argue by cases.

— Suppose that $h(a) \not\sqsubseteq b$. We let g map any $b' \in B$ to $\perp$ if $b' \sqsubseteq b$ and to $\top$ otherwise. Clearly g is strict and continuous. We let f be the composition of h and g (applied in this order), so f maps any $a' \in A$ to $\perp$ if $h(a') \sqsubseteq b$ and to $\top$ otherwise. Since h and g are strict and continuous, so is f. Moreover, whenever $h(a') = b'$, we have $f(a') = g(b')$, so $f\ \langle h\rangle^\top\ g$. On the other hand, $f(a) = \top$ while $g(b) = \perp$, so $f(a) \neq g(b)$.

— The case where $b \not\sqsubseteq h(a)$ is analogous. We let g map any $b' \in B$ to $\perp$ if $b' \sqsubseteq h(a)$ and to $\top$ otherwise. We let f be the composition of h and g (applied in this order), so f maps any $a' \in A$ to $\perp$ if $h(a') \sqsubseteq h(a)$ and to $\top$ otherwise. Again, g and f are strict and continuous, and $f\ \langle h\rangle^\top\ g$, but $f(a) = \perp$ while $g(b) = \top$, so $f(a) \neq g(b)$.

□

4. A characterization of $\top\top$-closed relations

When A and B are two complete partial orders and R is a relation between A and B, we let $\preceq_R$ be the least relation on the disjoint union $A \oplus B$ of A and B such that:

1 $R \subseteq \preceq_R$ and $R^{-1} \subseteq \preceq_R$,
2 $\sqsubseteq \subseteq \preceq_R$,
3 $\preceq_R$ is transitive,
4 $\preceq_R$ is admissible.

Condition 2 says that $\preceq_R$ extends the underlying partial order on A and B; it implies that $\preceq_R$ is reflexive. Because of conditions 2 and 3, the admissibility condition 4 is equivalent to the following:

4a. $\perp_A \preceq_R \perp_B$ and $\perp_B \preceq_R \perp_A$,
4b. for all chains $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \ldots$, if $x_i \preceq_R y$ for all i, then $\sqcup x_i \preceq_R y$.

There is always a least relation with the required properties, because the conditions are monotone. We may view these conditions as an inductive definition of $\preceq_R$.

It is always trivially the case that if $a\ R\ b$ then $a \preceq_R b \wedge b \preceq_R a$, but $a \preceq_R b \wedge b \preceq_R a$ does not necessarily imply $a\ R\ b$.

Theorem 2. Suppose that R is a relation between two complete partial orders A and B. Then R is $\top\top$-closed if and only if

$$R = \{(a, b) \in A \times B \mid a \preceq_R b \wedge b \preceq_R a\}$$

Proof. In order to establish one direction of this equivalence, we assume that R is $\top\top$-closed, $a \in A$, and $b \in B$, and show that if $a \preceq_R b$ and $b \preceq_R a$ then $a\ R\ b$. First, given f and g such that $f\ R^\top\ g$, we construct a relation T:

$$\begin{array}{rcl} T & = & \{(a, a') \in A \times A \mid f(a) \sqsubseteq f(a')\} \\ & & \cup \{(a, b) \in A \times B \mid f(a) \sqsubseteq g(b)\} \\ & & \cup \{(b, a) \in B \times A \mid g(b) \sqsubseteq f(a)\} \\ & & \cup \{(b, b') \in B \times B \mid g(b) \sqsubseteq g(b')\} \end{array}$$

This relation satisfies conditions 1–4 of the definition of $\preceq_R$. That is, T includes R, R^{-1}, and $\sqsubseteq$, and is transitive and admissible. (In checking the conditions, it is convenient to replace 4 with 4a and 4b.) Since $\preceq_R$ is the least relation that satisfies those conditions, we obtain $\preceq_R \subseteq T$. It immediately follows that if $f\ R^\top\ g$, then:

— if $a \preceq_R b$ then $f(a) \sqsubseteq g(b)$;
— if $b \preceq_R a$ then $g(b) \sqsubseteq f(a)$.

It further follows that if $f\ R^\top\ g$, $a \preceq_R b$, and $b \preceq_R a$, then $f(a) = g(b)$. In other words, if $a \preceq_R b$ and $b \preceq_R a$, then $f\ R^\top\ g$ implies $f(a) = g(b)$. Therefore, if $a \preceq_R b$ and $b \preceq_R a$, then $a\ R^{\top\top}\ b$, and hence $a\ R\ b$ by the $\top\top$-closure of R.

In order to establish the other direction of the equivalence of the theorem, we assume that $a \in A$, $b \in B$, $a \preceq_R b$, and $b \preceq_R a$ imply $a\ R\ b$, and prove that R is $\top\top$-closed. In order to do this, we must show that, for all $a \in A$ and $b \in B$, either $a\ R\ b$ or there exist f and g with $f\ R^\top\ g$ and $f(a) \neq g(b)$. So let us suppose that $a\ R\ b$ does not hold, and construct f and g with $f\ R^{\top}\ g$ and $f(a) \neq g(b)$. Since $a\ R\ b$ does not hold, one of $a \preceq_R b$ and $b \preceq_R a$ must be false. Let us assume that $a \preceq_R b$ is false; the other case is symmetric. We let f be the function that maps $a' \in A$ to $\top$ if $a' \not\preceq_R b$, and to $\perp$ otherwise. Similarly, we let g be the function that maps $b' \in B$ to $\top$ if $b' \not\preceq_R b$, and to $\perp$ otherwise. The functions f and g have the required properties, as we prove next.

— Since $a \not\preceq_R b$ but $b \preceq_R b$, we have $f(a) \neq g(b)$.
— The functions f and g are strict because $\perp_A \preceq_R b$ and $\perp_B \preceq_R b$ by conditions 2–4.
— In order to show that the functions f and g are continuous, it suffices to show that, whenever $x_0 \sqsubseteq x_1 \sqsubseteq \ldots$ is a chain, $\sqcup x_i \not\preceq_R b$ if and only if, for some i, $x_i \not\preceq_R b$. By conditions 2 and 3, $x_i \not\preceq_R b$ implies that $\sqcup x_i \not\preceq_R b$. Moreover, condition 4b says that if $\sqcup x_i \not\preceq_R b$ then $x_i \not\preceq_R b$ for some i.
— For all $a' \in A$ and $b' \in B$, if $a'\ R\ b'$ then $a' \preceq_R b$ if and only if $b' \preceq_R b$, because $a'\ R\ b'$ implies $a' \preceq_R b'$ and $b' \preceq_R a'$ (by condition 1), and by the transitivity of $\preceq_R$ (condition 3). Therefore, for all $a' \in A$ and $b' \in B$, if $a'\ R\ b'$ then $f(a') = g(b')$, so $f\ R^{\top}\ g$.

□

Corollary 3. Suppose that A and B are complete partial orders and that a relation $\preceq$ on $A \oplus B$ has the following properties:

1 $\sqsubseteq \subseteq \preceq$,
2 $\preceq$ is transitive,
3 $\preceq$ is admissible.

Let R be the relation between A and B defined by:

$$a\ R\ b \ \triangleq\ a \preceq b \wedge b \preceq a$$

Then R is $\top\top$-closed. Moreover, every $\top\top$-closed relation arises in this manner; that is, for every $\top\top$-closed relation R there is a relation $\preceq$ with properties 1, 2, and 3 and such that $a\ R\ b \equiv a \preceq b \wedge b \preceq a$.

Proof. The definition

$$a\ R\ b \ \triangleq\ a \preceq b \wedge b \preceq a$$

implies that $R \subseteq \preceq$ and $R^{-1} \subseteq \preceq$. Therefore, $\preceq_R \subseteq \preceq$, since $\preceq_R$ is the least relation with properties shared by $\preceq$. It follows that

$$a\ R\ b \ \equiv\ a \preceq_R b \wedge b \preceq_R a$$

and, by Theorem 2, that R is $\top\top$-closed.

The claim that every $\top\top$-closed relation arises in this manner is a straightforward consequence of Theorem 2: it suffices to let $\preceq$ be $\preceq_R$. □

There is a striking similarity between the conditions on $\preceq$ given in this corollary and the definitions of good partial preorders in (Abadi and Plotkin 1990) and of admissible preorders in (Simpson 1995). One apparent difference is that the corollary concerns a relation on the disjoint union of two complete partial orders, rather than a relation on a single partial order. Moreover, good partial preorders have an additional uniformity property and Simpson's admissible preorders are such that $x \preceq \bot$ only if $x = \bot$ (though this is not an essential requirement).

5. Comparison to admissible relations

The following two propositions concern some of the finitary properties of $\top\top$-closed relations.

Proposition 4. Suppose that R is a $\top\top$-closed relation. If $a\ R\ b$ and $a'\ R\ b'$ then $a\ R\ b'$ if and only if $a'\ R\ b$.

Proof. If $a\ R\ b$ and $a'\ R\ b'$ and $a\ R\ b'$ then $a' \preceq_R b$ and $b \preceq_R a'$, so $a'\ R\ b$ by Theorem 2. Symmetrically, if $a\ R\ b$ and $a'\ R\ b'$ and $a'\ R\ b$ then $a\ R\ b'$. $\square$

Proposition 5. Suppose that R is a $\top\top$-closed relation. If $a\ R\ b$ and $a''\ R\ b''$ and $a \sqsubseteq a' \sqsubseteq a''$ and $b'' \sqsubseteq b' \sqsubseteq b$ then $a'\ R\ b'$.

Proof. If $a\ R\ b$ and $a''\ R\ b''$ and $a \sqsubseteq a' \sqsubseteq a''$ and $b'' \sqsubseteq b' \sqsubseteq b$ then $a' \preceq_R b'$ and $b' \preceq_R a'$, so $a'\ R\ b'$ by Theorem 2. $\square$

The next two propositions compare admissible relations and $\top\top$-closed relations.

Proposition 6. Every $\top\top$-closed relation is admissible.

Proof. Suppose that R is $\top\top$-closed. Then $\bot_A \preceq_R \bot_B$ and $\bot_B \preceq_R \bot_A$ imply that $\bot_A\ R\ \bot_B$, by Theorem 2, so R is strict. Moreover, suppose that $a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots$ in A and $b_0 \sqsubseteq b_1 \sqsubseteq b_2 \ldots$ in B, and $a_i\ R\ b_i$ for all i; since $R \subseteq \preceq_R$ and $\preceq_R$ is inductive, $\sqcup a_i \preceq_R \sqcup b_i$; similarly, $\sqcup b_i \preceq_R \sqcup a_i$; hence $\sqcup a_i\ R \sqcup b_i$, by Theorem 2, so R is inductive. $\square$

Proposition 7. Not all admissible relations are $\top\top$-closed.

Proof. We give three examples of admissible relations that are not $\top\top$-closed. Any one of the three examples would suffice as proof, but they illustrate different points.

1. The first example shows that $\top\top$-closure can fail for simple finitary reasons. It is a variant of an example that Winskel suggested to Pitts.
   We let $A = B = \mathcal{O}$ and $R = \{(\bot,\bot),(\bot,\top),(\top,\top)\}$. All elements of A and B are related to one another by $\preceq_R$ in $A \oplus B$. Since $(\top,\bot) \notin R$, Theorem 2 implies that R is not $\top\top$-closed. In fact, R does not even have the property of $\top\top$-closed relations established in Proposition 4.
2. The second example is more complex, and illustrates some subtleties connected with infinite chains.
   We let
   $$A = \{\bot_A, a_0, a_1, a_2, \ldots, a\}$$
   where $\bot_A \sqsubseteq a_0 \sqsubseteq a_1 \sqsubseteq a_2 \ldots$ is a strictly increasing chain with least upper bound a;
   $$B = \{\bot_B, 0, 1, 2, \ldots, b, b'\}$$

where $\perp_B \sqsubseteq i \sqsubseteq b \sqsubseteq b'$ for every natural number i; and

$$R = \{(\perp_A, \perp_B), (a_0, 0), (a_1, 1), (a_2, 2), \ldots, (a, b')\}$$

We have $a_i \preceq_R i \preceq_R b$ for all i, so $a_i \preceq_R b$ for all i, so $a \preceq_R b$; conversely, we have $b \preceq_R b' \preceq_R a$, hence $b \preceq_R a$. By Theorem 2, $\top\top$-closure would require that we have $a \; R \; b$, but we do not.

3 In semantics, arbitrary relations are often a stepping stone for constructing symmetric and transitive relations, that is, pers. Pers are of particular importance because they commonly serve as the interpretations of types. The third example shows a per that is admissible but not $\top\top$-closed.
We let $A = \{\perp, a, a'\}$ with $\perp \sqsubseteq a \sqsubseteq a'$, and

$$R = \{(\perp, \perp), (\perp, a'), (a', \perp), (a', a'), (a, a)\}$$

The relation R is a per on A, and it is admissible. On the other hand, R does not have the property of $\top\top$-closed relations established in Proposition 5, because $\perp \; R \; a$ does not hold. Therefore, R is not $\top\top$-closed.

$\square$

6. Conclusion

The results of this paper are evidence of an intimate but nontrivial connection between $\top\top$-closure and admissibility.

— Every $\top\top$-closed relation is admissible, but not every admissible relation is $\top\top$-closed.
— Every $\top\top$-closed relation corresponds to an admissible preorder.

This connection helps keep some conceptual order. On the other hand, by itself, this connection does not shed much light on whether $\top\top$-closure or standard admissibility should be preferred as a basis for semantics and logics of programs. At this stage, it seems likely that the concept of $\top\top$-closure still requires generalization and elaboration. For example, admissibility makes sense for relations of any arity, while $\top\top$-closure has been defined only for binary relations. Indeed, it is not yet clear what should be the definition of a $\top\top$-closed unary relation. Further study of $\top\top$-closure may be worthwhile.

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